# Fractal Snowflakes

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- We often use numbers to count things.
- When we do it step by step, we use a scale of symbols $1, 2, \ldots$ to order the items we count one after the other.
- This process the application of a scale of numerals by a one-to-one correspondence.

- We handle cases where our count ends. Thus, we only use an initial segment of our scale, e.g. $1, 2, \ldots, n-1, n$.
- An application of an initial segment endows the collection of objects we have ordered with a numerical determination.
- This process makes use both of an ordered disposition of things and of a terminating point on the scale.

- If we decide to generalise numbers, it seems reasonable to rely on more terminating points and keep applying initial segments.
- The notation $1, 2, 3, \ldots$ identifies a limitation.
- We can apply initial segments only as long as they have a finite terminating point. We need more terminating points.

- We extend our notation as follows:

$$1, 2, 3, \ldots, ① - 1, ①, ① + 1, \ldots.$$

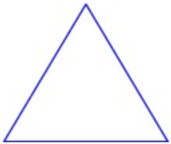
- Any term that contains ① (grossone) comes after all finite terms. In particular, the term ① marks off $\mathbb{N}$.
- We now think of $\mathbb{N}$ as the canonical completed sequence

$$1, 2, 3, \ldots, ① - 2, ① - 1, ①$$

# Contents

- We can classify sequential processes that involve infinitely many stages.
- If a process has features that vary at every stage, we can keep track of them after infinitely many, not only finitely many stages.
- We can study fractals that evolve from different starting points to different points at infinity, compute their area and perimeter.

# The Koch Snowflake (Helge von Koch 1904)



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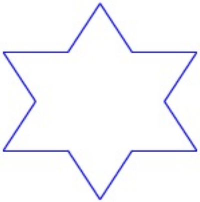

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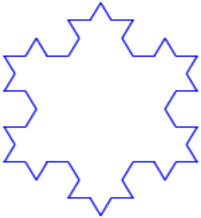

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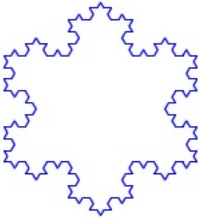

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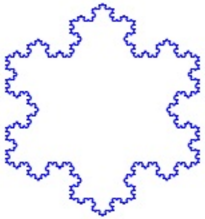

- Each figure is a stage in a construction with an equilateral triangle as an initiator. We may choose a different initiator.
- If two initiators are different, we may evolve them through the same number of stages. What happens?
- If this number is finite, we can tell that the resulting stages will be different. What happens after infinitely many stages?

- Instead of talking about differences, let us focus on one feature: the perimeter of a stage in fractal evolution.
- If we look at the result of finitely many stages from some initiator, we can compute its perimeter.
- Let us call $P_n$ the perimeter of the snowflake after $n$ stages. If $n$ only takes finite values, we have:

$$\lim_{n \to \infty} P_n = \infty$$

# Counting to (some) infinity

- If we adopt ①, we are in a position to look at distant stages.
- Because we are operating with a computational generalisation of finite numbers, we can in particular compute $P_{①}$.
- If we know the perimeters of two initiators, we can find out the perimeters of their ①-results and compare them.

- If the initiator is an equilateral triangle with sides of length $l$, its perimeter is $3l$.
- At the first iteration, each side is replaced by four sides. The length of each is one third of $l$.
- When we reach the second stage, there are $3 \cdot 4 = 12$ sides.
- At the next stage, there are $(3 \cdot 4) \cdot 4 = 48$ sides.

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- In these examples the perimeter increases. In general:

$$P_n = \frac{4^n}{3^{n-1}} l$$

- Note that this formula describes the evolution of the initiator at a terminating point $n$. We may set $n = ①$.

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- The difference between the perimeter of the ①-th and the preceding stage is infinitely large.
- Initiators may be infinitely significant. What about areas?

Read this paper:

Sergeyev, Ya. D. ‘The exact (up to infinitesimals) infinite perimeter of the Koch snowflake and its finite area’