1 Arithmetic with an infinite unit

Consider the set of natural numbers $\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$. If we collect together the first three elements we obtain $\{1, 2, 3\}$, where the largest number also measures the size of the set. This is true of the first four, five, $\ldots$, n elements. We assume that there is a measure ① for the whole set $\mathbb{N}$.
This means that $① > 1, ① > 2, ① > 3, \ldots$ and that ① is the largest natural number. Using ① (called *grossone*) we can now write the natural numbers like this:

$$\mathbb{N} = \{1, 2, 3, \ldots, ① - 2, ① - 1, ①\}$$

We assume that calculations with the symbol ① can be carried out in the familiar way. For example, the following equalities hold:
$① + ① = 2①$;
$① - ① = 0$;
$0① = 0$;
$① + 2① + 3 = 3① + 3 = 3(① + 1)$.

Calculate:

i. $① + 4① + 3$;

ii. $24(① + 3) - 8(9 + 3①)$.

We can use positive and negative numbers, too. For example, $2① - 3① = -①$ and $①(-2) = -2①$. Powers are also calculated as usual: for instance $3①(1 + ①) = 3① + 3①^2$.

Calculate:

iii. $①(2 + ①) - 4① - 3①(1 + ①)$;

iv. $3(①^2 - ① + 4) - 6①(① - 1) - 12$.

Finally, we can use the new symbol ① in calculations involving fractions. For instance, we can calculate:

$$\frac{①}{①} = 1,\ \frac{0}{①} = 0,\ \frac{①^2}{3①^2} = \frac{1}{3}.$$

Calculate:

v. $3\left(\frac{①^2}{2} + \frac{①^2}{3} - \frac{2①^2}{4}\right)$;

vi. $\left(\frac{①}{4} + \frac{1 - ①}{2①}\right) - \frac{① - 2}{4}$.

2 Sizes of parts of $\mathbb{N}$

The number ① measures the size of $\mathbb{N}$. We would like to say that, if we take away from $\mathbb{N}$ all of the odd numbers, then we are left with half of ① numbers, i.e. we are left with $\frac{①}{2}$ numbers. This should be the size of the set of even numbers. Note that we can write this set as:

$$\{2, 2+2, 2+4, 2+6, \ldots\} = \{2, 2+1\cdot 2, 2+2\cdot 2, 2+2\cdot 3, \ldots\}$$

which is a special case of the general progression:

$$\{k, k+n, k+2n, k+3n, \ldots\}$$

when $k = 2$ and $n = 2$. We call the general progression $\mathbb{N}_{k,n}$. Thus, the set of even numbers is $\mathbb{N}_{2,2}$. Whenever we can write a set of numbers in the form $\mathbb{N}_{k,n}$ we assume that its size is $①/n$. **Since a set is a collection of objects, $\frac{①}{n}$ is a natural number that counts them**.

i. Show that $\mathbb{N}_{1,2}$ is the set of odd numbers. For which values of k, n can we get the set of even numbers?

ii. Write down the first five elements of $\mathbb{N}_{3,5}$ and $\mathbb{N}_{7,7}$.

iii. What values of k, n determine the set of multiples of 3? What is the size of this set?

3 Even and odd numbers

A whole number is even if, and only if, it is a multiple of 2. Otherwise it is odd. We have seen in section 2 that $\frac{①}{2}$ is a natural number. Thus, $① = 2\frac{①}{2}$ is even. It follows from this that $① - 1$ is odd (even and odd numbers appear consecutively).

i. Is $\frac{①}{3}$ even or odd?

ii. Is $2① + 1$ even or odd?

iii. Is $① - \frac{①}{6}$ even or odd?

iv. Is $\left(\frac{①}{5} - \frac{①}{7}\right) + \frac{①}{2}$ even or odd?

4 Geometric Series

Consider the following sequence of numbers:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

It is usually assumed that there are as many terms in this sequence as there are numbers in $\mathbb{N}$. This means that the sequence has ① terms. **The summation of all ① terms in the sequence is called the geometric series of ratio** $1/2$**.**

i. Write the last three terms of the summation.

ii. Let the value of the summation be s: verify that $2s = 1 + s - 1/2^{①}$.

iii. Solve the previous equation for s and thus find the sum of the series.

iv. How close is this sum to 1?

5 Thomson's Lamp without ①

A lamp, which is initially off, is switched on after $1/2$ minutes, off after $1/2 + 1/4$ minutes, then on again after $1/2 + 1/4 + 1/8$ minutes, and so on. We assume that the lamp is switched on and off as many times as there are numbers in $\mathbb{N}$.
If we try to describe Thomson's lamp without using the infinite unit ①, we can represent the sequence of switchings as follows:

$$\mathsf{on}_1, \mathsf{off}_2, \mathsf{on}_3, \mathsf{off}_4 \ldots.$$

We can also deduce that, for each natural number n, the n-th switching has taken place before 1 minute has passed.

i. How long does it take to switch the lamp on and off 4 times?

ii. How long does it take to switch the lamp on and off 10 times?

iii. There is one question which seems really hard to answer without appealing to ①. The question is: when 1 minute has passed, is the lamp on or off?

iv. Consider the following argument:

> If the lamp is on before one minute has passed, then at a later time, before one minute has passed, the lamp is turned off. On the other hand, if the lamp is off before one minute has passed, then at a later time, before one minute, the lamp is turned on. Therefore, after one minute has passed, the lamp is neither on nor off, since each one of these states is turned into the other before one minute has passed.

Discuss this argument. Do you find it compelling?

6 Thomson's Lamp with ①

Now let us tackle Thomson's lamp using the infinite unit ①.

i. How long does it take to switch the lamp on and off $①/2$ times?

ii. How long does it take to do so ① times?

iii. If we count the act of switching the lamp on or off, then the even-numbered acts of switching turn the lamp off. Is the lamp on or off after 3 switchings?

iv. Is the lamp on or off after 237 switchings? Is it on or off after $①/3$ switchings? What about $2① + 1$ switchings?

v. Is the lamp on or off after one minute has passed?

References

Sergeyev, Ya.D. *The Arithmetic of Infinity*. Rende: Orizzonti Meridionali (available as an e-book on Kindle).