First Steps in the Arithmetic of Infinity

a new way of counting and measuring

Davide Rizza

University of East Anglia

Indice

1. Introduction

1.1 Aims of this textbook

This textbook is designed to allow students and secondary school teachers to explore, in elementary terms, a fascinating extension of the practices of reckoning and computing to infinitely large and infinitely small orders of magnitude. The extension, known as Arithmetic of Infinity, was introduced by Yaroslav Sergeyev[1] in 2003 and progressively developed in a series of papers, notably [**25**, **26**, **27**, **28**, **32**]. The increasing significance of this approach has subsequently become apparent in several central fields of applied mathematics like numerical analysis (see e.g. [**1**, **17**, **31**]), operations research (recently, in [**8**, **9**]), computability theory (see especially [**29**, **30**]), the study of cellular automata (for instance [**6**, **7**]) and probability theory (see [**23**, **24**]).

The Arithmetic of Infinity is based on a reconstruction of the basic ideas of computation and counting or reckoning, which naturally leads to refinements of known mathematical results or to new results altogether. This fact, together with a distinctive accessibility of its motivation and ideas, makes the Arithmetic of Infinity remarkably fruitful in the context of school teaching. On the one hand, its adoption opens up a new perspective on what is already familiar, showing how it can be developed beyond the bounds of the standard curriculum. On the other hand, its articulation points to new

[1] http://wwwinfo.deis.unical.it/yaro/.

results and areas of application, which the interested student is invited to pursue independently, as far as their interests go (the six guided exploration of paradoxes included in this textbook provide as many invitations to do so).

Because of this, students engaging with the Arithmetic of Infinity are enabled both to develop a keener sense for the application of mathematical ideas to new problems and a more explicit awareness of the creative character of mathematical thinking. The latter is often difficult to appreciate when the routine encounters with mathematics mostly involve rote exercises and a deliberate focus upon an artificially restricted set of given techniques. Learning the Arithmetic of Infinity is a way of realising that, even at a basic level, mathematical concepts may be modified and new techniques may be introduced, in response to the terms of the problems at hand. A picture of mathematical knowledge as an immense repository of set and indisputable dogma is replaced by the more realistic portrait of an array of problem-solving instruments that can be enriched and altered in response to actual challenges.

The author has been able to make use of preliminary drafts of the materials contained in this text in the context of several school workshops, conducted in Italy and the United Kingdom (some of the responses from teachers and students are recorded on www.numericalinfinities.com). These activities have generated results of interest to current research in mathematics education (see [**10**]). A need was felt to try and increase the accessibility of teaching materials that had proved successful and could be presented after several refinements made possible by much helpful feedback from the workshop participants.

1.2 How to use this textbook

The textbook is split into two parts. The first part motivates and introduces basic techniques of computation within the Arithmetic of Infinity (Chapter 2), in order to apply them (Chapter 3) to infinite sequences and infinite series. The second part is made of six entirely independent exercise-based explorations of paradoxes of infinity. All explorations are self-contained and can be carried out without having read any other part of the textbook (this why the same material is repeated at the beginning of each).

On account of its structure, the textbook can be used in at least three distinct ways:

- **Option 1**: the teacher might set up joint study sessions of Chapters 2 and 3 with the whole class. The sessions revolve around working through all set exercises. At this point, the teacher may assign to individual students or groups of students the autonomous study of a paradox of infinity. The task of the student or group is then to present what they have learned about the paradox to their peers.
- **Option 2**: the teacher gains some background knowledge of the Arithmetic of Infinity by studying Chapters 2 and 3. On its basis, the teachers organises one or more workshops on paradoxes of infinity, making selected worksheets from the second part of this textbook available to students. Since each worksheet is self-contained, there is no need for students to have a deeper background, but they may be guided by the teacher into the exploration of the problems proposed in whichever worksheet they are assigned.
- **Option 3**: the teacher may freely organise a structured activity revolving around one or more paradoxes of infinity, using the worksheets from the second part of the textbook as a springboard to design the activity itself. This only requires study of selected worksheets, but the teacher may well rely on Chapters 2 and 3 in order to achieve a firmer grasp of key ideas.

A booklet with solutions to all exercises is available in pdf format upon request. It suffices to send an email to d.rizza@uea.ac.uk.

1.3 Beyond first steps

The reader of this textbook may wish to gain further knowledge of Sergeyev's approach to computation and counting with infinitely small and infinitely large quantities. To this end, the very readable book *Arithmetic of Infinity* (2003, Kindle Edition 2013), written by Sergeyev himself, provides a good source. For the more advanced reader, the website

http://theinfinitycomputer.com/arithmetic.html

contains a full list of the research papers on the Arithmetic of Infinity published to date. Two of these publications may be of interest to the non-specialist:

- Rizza, D. (2018) 'A study of mathematical determination through Bertrand's paradox', *Philosophia Mathematica* 26, pp.375–395.

- Sergeyev Ya.D. (2016), ‘The exact (up to infinitesimals) infinite perimeter of the Koch snowflake and its finite area’, *Communications in Nonlinear Science and Numerical Simulation*, 31, 21–29.

An extensive survey of almost fifteen years of work on the Arithmetic of Infinity is provided by:

- Sergeyev, Ya. D. (2017) ‘Numerical infinities and infinitesimals: Methodology, applications, and repercussions on two Hilbert problems’, *EMS Surveys in Mathematical Sciences*, 4, 219–320.

1.4 Comments and feedback

This textbook has resulted from continuous interaction with students and teachers. It is hoped that it can undergo future revision and expansion in light of growing adoption in educational contexts. Comments and feedback from those who have made use of it are welcome and will provide helpful pointers to further improvement.

2. Arithmetic with the infinite unit ①

2.1 Numeral stores and numeral systems

If we can count a bunch of objects, we are able to determine an alignment of numerical symbols that achieves a specifiable terminal or end stage after labelling each one of the objects we wished to consider. If, for instance, we were interested in a bunch that happened to contain exactly nine distinct objects, then a completed count of them would lead to the following alignment of numerical symbols:

$$1,2,3,4,5,6,7,8,9.$$

Such an alignment is in effect a labelling, since we may decide to 'name' each one of the objects considered by the numerical symbol assgned it. Numerical symbols are also known as *numerals*[1]. Now let us for the moment suppose that we have at our disposal only a limited store of numerals, including exactly the symbols $1,2,3,4,5,6,7,8,9$. With such a store in hand, we are in a position to produce an alignment for each bunch of nine or fewer items, but we are bound to run into troubles when larger bunches occur to our consideration. In the absence of a more extensive store of numerals, we are not however in a position to give up evaluating sizes of larger bunches. We may still offer evaluations, but these are inevitably inaccurate, since, in a large bunch, a numerical alignment ends before we

[1] Thus, an equivalent alignment to the one just illustrated could have been produced using the Roman numerals I, II, III, IV, V, VI, VII, VIII, IX.

review every object of interest. A bunch of eleven object is, from this point of view, equivalent to a bunch of one hundred objects: each forces us to use up our numeral store without producing a complete labelling of the objects we seek to count. If we work with a numeral store characterised by limited capabilities, we may supplement it with an additional symbol that enables use to indicate a situation in which our 'ordinary' numerals are not enough to produce a completed count. Let the additional symbol be ∞, which, in the present context, really means 'more than nine'. If we handle a bunch of ten objects, we can then assign it the approximate evaluation ∞. Adding one distinct object to the bunch of ten does leave the evaluation ∞ intact, since we identify situations in which we run out of numerical resources. Note that, if we tried to express our last two evaluations in terms of arithmetical calculations, we would generate the equality $\infty + 1 = \infty$. Such an equality does not express a property of very large orders of magnitude but a property of relatively small numeral stores. Moreover, if we dealt with two distinct bunches, of ten and eleven objects respectively, we would have to evaluate each of them at ∞ and we could express the result of adjoining them by the arithmetical equality $\infty + \infty = \infty$. If we adjoin more items to an amount of items that already exceeds our bounded resources, we can only confirms that our bounds are exceeded. The last remarks are meant to illustrate the fact that, if we wished to use the numeral store:

$$1,2,3,4,5,6,7,8,9,\infty$$

in applied calculations, we might be able to proceed in the ordinary manner when dealing with sufficiently small numbers, but we would be forced to abandon familiar arithmetical rules when engaging in calculations that involve large bunches of items. In such cases, we may have to resort to equalities like $\infty + 1 = \infty$ e $\infty + \infty = \infty$. From an applied perspectives, the last equalities describe conditions under which counting is carried out with a low accuracy, due to limitations affecting the available numeral store.

Low accuracy carries two distinctive problems: first, as we have just seen, it allows us to carry out arithmetical evaluations that may be too coarse to be serviceable (we might want to be able to introduce a sharp distinction between a bunch of ten items and a bunch of eleven items). Second, it prevents us from carrying out certain arithmetical evaluations. In an applied context, sharp arithmetical evaluations are helpful because they enable us to anticipate the consequences of our actions. To clarify with a trivial example, if we hold three apples and eat one, the result of this is subsequent possession of only two apples. We clearly do not need to actually eat an apple to anticipate how many will be left, if we initially have three: we simply carry out the subtraction $3 - 1 = 2$. It is something that even our limited numeral store enables us to do. When, on the other hand, we confront a very large bunch of items, we may not be able to determine, by an anticipatory calculation, any uniquely determined value (even approximately) to the result of subtracting many items from it. The calculation in question will be expressed by the term $\infty - \infty$, which does not allow any further determination. From an applied point of view, we cannot use this arithmetical term to anticipate any particular consequence in reasoning. In particular, for each symbol in our numeral store, there is an applied context in which $\infty - \infty$ is evaluated at that symbol (if, for instance, we discarded ten out of twenty items, we would describe the result by $\infty - \infty = \infty$, but if we were to discard two items out of eleven, we should express the result as $\infty - \infty = 9$. Discarding nine out of eleven items, by contrast, should lead to $\infty - \infty = 2$). We say that the term $\infty - \infty$ is an **indeterminate form**. An immediate result of this discussion is that, in any applied context in which typical bunches contain at least ten items, the numeral store $1,2,3,4,5,6,7,8,9,\infty$ should prove highly inadequate. We would run into troubles when seeking to control in calculations the consequences of actual operations on the bunches we are presented with.

The general lesson that may be learned from the foregoing observations is that reckoning and carrying out calculations with bounded resources, in particular a bounded numeral store, leads to two distinctive problems. In the first place, the given numeral store restricts our ability to assign sharp numerical discriminations to distinct sizes. Secondly, the given numeral store gives rise to indeterminate forms, which effectively restrict the application of the arithmetic with bounded resources we are working with. If we think of a numeral store as an instrument designed for application to concrete contexts, we see that a bounded numeral store like the one described so far is an instrument of limited serviceability, surely inadequate to frequently encountered concrete contexts. Whenever we deal with instruments in some respects inadequate, their shortcomings immediately suggest the possibility of replacing them by better instruments. An immediate improvement at our disposal is adopting an enlarged numeral store. This solution, if carried out in a primitive manner, i.e. through the addition of more distinct symbols, simply shifts the problems highlighted so far, without resolving them. If, for example, we decided to extend our numeral store to one hundred distinct symbols, we would not be rid of inaccurate evaluations and indeterminate forms.

An abstract way of getting around this problem (which does not vanish in practice, on account of technological limitations on computationally manageable orders of magnitude) is to think not of an initially fixed numeral store, but of a systematic way of extending our numeral resources as needed. We work with a *numeral system*.

The best known numeral system is in all likelihood the one in base ten. Its backbone is provided by the sequence of ten symbols:

$$0,1,2,3,4,5,6,7,8,9$$

which are used as indices of the powers of ten. This is to say that, for example, we use the numeral 2 as an index of the term $2 \cdot 10^0$. In an analogous fashion, the numeral 27 indicates the term $2 \cdot 10^1 7 \cdot 10^0$, which in turn designates the number $20+7=27$. Proceeding along the same lines, we take 584 to indicate the term:

$$5 \cdot 10^2 8 \cdot 10^1 4 \cdot 10^0,$$

which in turn designates the number $500+80+4=584$. The main advantage of using a numeral system like the one just described is its flexibility. Numeral stores like the one used above can be inserted as initial segments within an extended environment in base ten. We normally use dots to indicate the possibility of insertion. Thus, our initial store of nine symbols may be inserted in the numeral system in base ten thus:

$$1,2,3,4,5,6,7,8,9,\ldots.$$

An enlargement of the nine-symbol store to a store with one hundred symbol could similarly be inserted within the base ten numeral system as follows:

$$1,2,3,\ldots,50,51,52,\ldots,98,99,100.$$

Our earlier discussion now leads to a definite question: even though numeral systems improve upon numeral stores, are numeral systems affected by similar limitations?

2.2 Natural numbers

The numeral system in base ten we have just described is an abstract device through which we may regard some fixed numeral stores as extensible ones. A feature that numeral stores and this numeral system share is that they are intended for application to very diverse contexts. There are many different things that we may want to count: among them are the symbols of a particular numeral store or system. For instance, we can use a numeral store of nine symbols to count the symbols $1,2,3$. We can certainly use the numeral system in base ten to the same end. A less straightforward situation arises when we consider the task of counting the symbols of a numeral system. It is not difficult to count a few of them. If presented with the numerals:

$$5,6,7,8,$$

we could easily produce an alignment with the count $1,2,3,4$. If, more generally, we were given a sequence of inscriptions of the form:

$$1,2,3,4,\ldots,n-1,n,$$

with n exprssible in base ten, our numeral system would still allow us to count the given inscriptions. We may, in effect, think of an application of the numeral system as a kind of tape measurement, where we work with a tape $1,2,3,\ldots$ along which sequences of inscriptions like $1,2,\ldots,n-1,n$ admit an alignment. In classical mathematics, the abstract tape we are using is treated as an object namely the infinite set $\mathbb{N}$. We describe this object as follows:

$$\mathbb{N} = \{\mathbf{1},\mathbf{2},\mathbf{3},\ldots\}.$$

where the symbols in bold are intended to identify items in the abstract bunch $\mathbb{N}$, which are trivially related to signs from the numeral system in base ten. Curly brackets indicate that we look at **1** o **2** not in isolation, but as special items within a specific collection called $\mathbb{N}$. We call $\mathbb{N}$ an infinite set. One classical way of spelling out 'infinite' is to say that no sequence o the form $1,2,\ldots,n-1,n$, with n expressible in base ten, affords a completed count of $\mathbb{N}$. We may conjecture that, as long as we handle the numeral system in base ten, we can count and calculate over sufficiently short sequences from $\mathbb{N}$, but we are going to encounter problems if we seek to extend these techniques to the whole of $\mathbb{N}$. It is not unreasonable to consider the whole of $\mathbb{N}$: it is, according to its standard treatment, conceived as a given collection of a specifiable size. We adopt the following principle concerning sizes.

Euclidean[2] **Size**: the whole is greater than the part.

We can now show that the inaccuracies arising from the application of a relatively small numeral store to contexts in which typical collections are comparatively very large are replicated by the application of the numeral system in base ten to $\mathbb{N}$ and its infinite parts. Next, we show that the same kind of application produces indeterminate forms. In order to reach both conclusions, we introduce what may be roughly seen as a counterpart to the symbol ∞, which we had adopted to discuss numeral stores. The symbol in question, due to Georg Cantor (1845–1918), is $\aleph_0$ ('aleph zero'), which we associate with $\mathbb{N}$ as a measure of its size (certainly large, if compared to the size of a sequence of the form $1,2,\ldots,n-1,n$, which omits most numbers in $\mathbb{N}$). In line with Cantor's ideas, if C is a collection of items from $\mathbb{N}$, we take C to have size $\aleph_0$ if it is possible to transform a count of $\mathbb{N}$ in base ten into a count of C in base ten. A count of $\mathbb{N}$ of the kind we allow looks like this:

$$1,2,3,4,5,6,\ldots n-1,n,n+1,\ldots,$$

where n is expressible in base ten. For the sake of illustration, let us consider the collection of even numbers. Given the count of $\mathbb{N}$ just described, we are in a position to transform it into a count of the collection of even numbers by doubling each term of the original count. We obtain the correspondence:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \ldots \\ 2 & 4 & 6 & 8 & \ldots. \end{array}$$

Our criterion for size comparison entitles us to assign $\aleph_0$ to the collection of even numbers. If we took the predecessors of the even numbers, we could transform a count of $\mathbb{N}$ into a count of the collection of odd numbers, thus:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \ldots \\ 1 & 3 & 5 & 7 & \ldots. \end{array}$$

[2]From the name of the Greek mathematician Euclid (III-IV century B.C.), who included this principle among the Common Notions in his *Elements*.

The size of last collection is therefore $\aleph_0$.

Exercise 1. Verify that the following collections have size $\aleph_0$:

a) the collection obtained by deleting **1** and **2** from $\mathbb{N}$;
b) the collection obtained by deleting the numbers between **1** and **100** (endpoints included) from $\mathbb{N}$;
c) the collection of multiples of **3** in $\mathbb{N}$;
d) the collection of multiples of **4** in $\mathbb{N}$;
e) the collection obtained from the collection of the even numbers by deleting all multiples of **4**.

The collection obtained from $\mathbb{N}$ by deleting **1** and **2** omits two items from the whole. It is therefore a strict part[3], which, by Euclidean Size, should be smaller than the whole. Our criterion for the assignment of $\aleph_0$ as a measure of size, which depends on the selection of a specific numeral system, prevents us from providing a numerical discrimination between whole and strict part. This issue stands out even more strikingly when we delete infinitely many items from $\mathbb{N}$, omitting e.g. all odd numbers, and reach the evaluation $\aleph_0$ for the size of the remainder. The problem persists even when we delete infinitely many items (e.g. the multiples of four, as we saw in exercise 1.d) from the result of a previous deletion like the collection of the even numbers in $\mathbb{N}$. In short, $\aleph_0$ behaves like ∞ with respect to the application of a numeral store. Indeterminate forms are to be expected. In full analogy with what we have already observed, a natural instance is the term $\aleph_0 - \aleph_0$. As we shift contexts, this term may be freely associated with an infinitely large range of evaluations, including $\aleph_0$ as well as any fixed n fissato expressible in base ten. For instance, if we delete from $\mathbb{N}$ the collection described in Exercise 1.a, we may expect $\aleph_0 - \aleph_0 = 2$ but, when we delete the multiples of four from the collection of the even numbers in $\mathbb{N}$, we have to switch our expectation to $\aleph_0 - \aleph_0 = \aleph_0$. We cannot exercise any arithmetical control over the term $\aleph_0 - \aleph_0$. This is not to say that the very employment of $\aleph_0$ should be regarded as inconsistent, but only to point out that it is inconsistent with the arithmetical demand to introduce numerical resources that satisfy the principle of Euclidean Size. This principle is important because the arithmetical techniques that satisfy

[3] We take the whole to be, in a loose sense, a part of itself. We are, however, always interested in discriminating strict parts of a collection.

it enable systematic discriminations of size and prevent the appearance of computational obstacles like indeterminate forms or computational shortcomings like approximate evaluations. Our goal is now clear: to develop arithmetical techniques that enable us to work with $\mathbb{N}$ and its parts without giving rise to the difficulties produced by the attempt at making computational use of $\aleph_0$. We have pointed out that the problems affecting the choice of a fixed numeral store may be tackled by selecting a richer numeral store. We solve the problems affecting the choice of a fixed numeral system (the one in base ten) by selecting a richer numeral system.

2.3 A new numeral system

The numeral system in base ten introduced earlier was compared to a tape that allows us to produce a completed count of items as an alignment of numerals $1,2,\ldots,n-1,n$ (with n expressible in base ten) included in a more extensive environment, i.e. the rest of the tape stretching farther than the terminal numeral achieved upon completion of a count. In order to obtain a numeral system richer than the one in base ten, we proceed in a similar manner: we work with a more extensive tape, such that a count of the entire collection $\mathbb{N}$ is included in the tape as an initial segment reaching a terminal numeral. The terminal numeral for a completed count of $\mathbb{N}$ is ① ('gross-one'). Our new measuring tape may be described as follows:

$$1,2,3,\ldots,n-1,n,\ldots,①-1,\ ①\ ,①+1,\ldots.$$

with n expressible in base ten. We take a completed count of $\mathbb{N}$ to produce the completed alignment:

$$1,2,3,\ldots,①-2,①-1,①.$$

This alignment is included in the new measuring tape, since the latter proceeds to list $①+1,①+2$ etc. after the symbol ①. In order to make good arithmetical use of the new measuring tape, we introduce two postulates. The first is that ① should be located after each numeral expressible in base ten. The second postulate is that the familiar arithmetical properties of operations like addition or multiplication should extend to ① and numerical terms containing it (e.g. $①-1,①+1$ etc.). Following the introduction of the set $\mathbb{N}$ on the basis of an abstract considerations of numeral expressible in base ten, we may now introduce an extended collection corresponding to the new numeral system we have introduce. This extended collection is:

$$\mathcal{N} = \{\mathbf{1}, \mathbf{2}, \ldots, ① \textbf{ - } \mathbf{1},\ ①\ , ① \textbf{ + } \mathbf{1}, \ldots\}$$

and, with respect to $\mathcal{N}$, we may state our two postulates as follows:

- **Postulate 1**: if n is a positive number expressible in base ten, the inequality $n < ①$ holds;
- **Postulate 2**: The objects in $\mathcal{N}$ satisfy the formal properties of order, addition, multiplication and exponentiation that holds in $\mathbb{N}$.

We illustrate how the postulates can be used: consider a number m expressible in base ten. Then $m+1$ is also expressible in base ten and $1 < m+1$. By postulate 1, we have $m+1 < ①$. By postulate 2, the formal property:

$$\text{if } m+1 < x \text{ then } m < x-1$$

holds in $\mathcal{N}$. In particular, it holds when we set $x = ①$. There follows $m < ① - 1$. Since m is an arbitrary number expressible in base ten, we have deduced that an analogue of Postulate 1 is satisfied by $① - 1$. Moreover, since the inequality:

$$x - 1 < x$$

holds in $\mathcal{N}$ by Postulate 2, we can deduce $① - 1 < ①$.

Exercise 2. Use Postulates 1 and 2 in order to deduce the following inequalities:

a) $① < ① + 1$;

b) $① < ① + ①$;

c) $2① < 6①$;

d) $7(① - 1) < ①^2$ (N.B.: $①^2$ is an abbreviation of the product $① \cdot ①$).

We have pointed out that the numeral system in base ten makes use its basic symbols as indice of powers of ten and that these indices are simply juxtaposed to indicate the sums of corresponding powers of ten. For instance, the string of indices 123 is a numeral in base ten that indicates the record:

$$1 \cdot 10^2 2 \cdot 10^1 3 \cdot 10^0,$$

where each index denotes a number smaller than the base. Our new numeral system has base ①. Its terms can be written using a notation entirely analogous to that adopted for base ten numerals when we make the powers of the base explicit. We shall not make any extensive use of this notation, but we devote a short subsection to it, for the sake of completeness.

2.3.1 Notation in base ①

We are able to represent a number in base gross-one using indices smaller than gross-one. This implies that all base ten numerals are indices. For instance, the numeral term $① + 2$ corresponds to the base ① record:

$$1 \cdot ①^1 2 \cdot ①^0,$$

whwre $①^0 = 1$ by Postulate 2. Clearly, if n is expressible in base ten, its base ① record is:

$$n \cdot ①^0.$$

This clearly shows in what way the numeral system in base ① is an extension of the system in base ten. The latter is a special case of the former, in which the only power of ① that can be allowed is the zeroth power. In the context of $\mathscr{N}$, we only admit records with positive or zero powers of ①. Negative powers, and a more extensive numerical notation, will be introduced later. For the moment, we may adopt the provisional distinction between *finite* records relative to $\mathscr{N}$, in which no positive powers of ① occur, and *infinite* records relative to $\mathscr{N}$, in which positive powers of ① do occur. Thus, for instance:

$$1 \cdot ①^2 \text{ and } 2 \cdot ①^7 5 \cdot ①^3 12 \cdot ①^0$$

are infinite records. It is worth remarking that no restriction is placed on the exponents of ①, which may vary within $\mathscr{N}$ (their range of variation will be extended later). In this vein, $1 \cdot ①^{①}$ is an infinite term.

Exercise 3. Write the following terms as ① records:

a) 10;

b) 487;

c) $① + 6$;

d) $5① + 3$;

e) $7①^3 + 51① + 234$.

f) $①^{①+3}$.

From now on we shall abandon base ① records and we shall only make use of arithmetical terms, which which it is easy enough to carry out calculations.

Example 1. *Simplify the expression* $5(①+2)-4(①+3)$. Ordinary arithmetical calculations yield:

$$\begin{aligned} 5(①+2)-4(①+3) &= (5①+10)-4(①+3) \\ &= (5①+10)-4①-12 \\ &= 5①+10-4①-12 \\ &= 5①-4①+10-12 \\ &= ①-2 \end{aligned}$$

Exercise 4. Simplify the following terms:

a) $①+4①+3$;

b) $3①+2(①+3)$;

c) $5(①+4)+4(①+5)$.

2.4 Integers

So far we have developed arithmetical resources that would allow us to study $\mathbb{N}$ inside the richer environment $\mathcal{N}$, which satisfies Postulates 1 and 2. We did this primarily because we wished to develop a more powerful arithmetical instrument and increase our computational power. If we stopped at the stage just reached, we would not be in possession of an arithmetical framework equipped to deal with terms like $1-3$ or $①-①$, which are not defined in $\mathcal{N}$. To this end, we consider an extension of $\mathcal{N}$ to a wider environment in which we deal with signed numbers and Postulates 1 and 2 continue to hold. More precisely, Postulate 1 remains unchanged, but Postulate 2 must be reformulated with respect to $\mathbb{Z}$ (the set of zero, positive and negative integers) and the following collection:

$$\mathcal{Z} = \{\ldots -(\textbf{① +1}),\ -\textbf{①}\ , -(\textbf{① - 1}), \ldots, -\mathbf{2}, -\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \textbf{① - 1},\ \textbf{①}\ , \textbf{① + 1}, \ldots\}.$$

Suppose that we wish to compute $①-①$. By the new form of Postulate 2, the relation $x-x=0$ holds in $\mathcal{Z}$, thus $①-①=0$. By the same clue, given the term $2① - 3①$, the relation $2x - 3x = -x$ can be invoked to conclude $2① - 3① = -①$. A final illustration: since any product in which a factor is zero equals zero, we can deduce $0 \cdot ① = 0$ from Postulate 2. A straightforward arithmetic over $\mathscr{L}$ is now available.

Example 2. *Simplify the term* $3①(2+3①) - ①(3-①)$. We proceed in the usual way:

$$\begin{aligned} 3①(2+3①) - ①(3-①) &= (6① + 9①^2) - ①(3-①) \\ &= (6① + 9①^2) - 3① + ①^2 \\ &= 6① + 9①^2 - 3① + ①^2 \\ &= 10①^2 + 3① \\ &= ①(10① + 3) \end{aligned}$$

Exercise 5. Simplify the following terms:

a) $24(① + 3) - 8(9 + 3①)$;
b) $10(6(3① - 4) - 2(4① + 7))$;
c) $①(2 + ①) - 4① - 3①(1 + ①)$;
d) $①[3(①^2 - ① + 4) - 6①(① - 1) - 12]$;
e) $2[①(2① + 7) + 2(①^2 + 1) - ①(①^2 + 4① - 11)]$;
f) $(① + 1)^3 - ①(3① + 3)$;
g) $2 - 2① + [(① + 1)(① - 1)]$.

2.5 Measures

We have in hand a mathematical setup designed to enable calculations with finite and infinitely large, signed quantities. Our interest, however, is not in arithmetical calculations as such, but in applied arithmetic, i.e. in the connection between counting and computing. The numeral system in base ① is intended to afford a computational improvement over $\aleph_0$ in connection with counting certain special collections, namely finite and infinite parts of $\mathbb{N}$. Our work from section 2.3 allows us to provide a completed count of $\mathbb{N}$ and, as we shall see in a moment, to discriminate this count from others that, in base ten, yield the same evaluation, namely $\aleph_0$. In short, we can vindicate the principle of Euclidean Size. Pushing the analogy between numeral systems and measuring tapes slightly further, we can say that, in our new numerical notation, we are able to carry out tape measurements of finite and infinite collections of items. The readings of each measurement, i.e. the numerical symbol that terminate completed counts, is the measure of a collection. Thus, we may, in an applied vein, think of counting items as a form of measurement effected against a specific instruments that provides numeral readings. For instance, a completed count of $\mathbb{N}$ produces an alignment of numerals starting with 1 and reaching the terminal symbol ①, which we may take to be the measure of $\mathbb{N}$. We shall at times refer to collections for which we can specify a terminal numeral reading as *measured* collections. In this terminology $\mathbb{N}$ is measured and its measure is ①.

With reference to $\mathscr{X}$, we are easily able to determine further measured collections. If, for instance, we add the item **0** to $\mathbb{N}$ we can carry out a completed count starting with 1 whose numeral reading is ① $+ 1$. The collection of non-negative integers is measured and its measure is ① $+ 1$. However obvious it might seem, it is important to realise that we obtain ① $+ 1$ by summing the measure of $\mathbb{N}$ e the measure of the one-item collection exhausted by **0**. This is a simple illustration of the general fact that our arithmetic is a calculus of measures, or an applied arithmetic. It is designed to transpose the result of tape measurements into the results of arithmetical computations. A few examples will clarify the last remark.

Example 3. The collection $\mathbb{Z}$ is obtained by combining the collection $\mathbb{N}$, the one-item collection exhausted by **0** and the collection $-\mathbb{N}$, which contains each item in $\mathbb{N}$ preceded by the minus sign. The first collection has measure ①, the second has measure 1 and the third has measure ① (why?). There follows that $\mathbb{Z}$ has measure ① $+ 1 +$ ① $= 2$① $+ 1$.

Example 4. The collection X obtained from $\mathbb{Z}$ by deleting $-\mathbb{N}$ and the even numbers smaller or equal to 100 is measured and has measure ① $- 50$. We know that $\mathbb{Z}$ has measure 2① $+ 1$ and that $\mathbb{N}$ has measure ①. The collection of even numbers smaller or equal to 100 in $\mathbb{N}$ is 50. Finally, the measure of the one-item collection exhausted by **0** us 1. The measure of X is computed as follows: (2① $+ 1) - ($① $+ 50 + 1) =$ ① $- 50$.

Exercise 6. Compute the measures of the following collections included in $\mathscr{Z}$:

a) the collection A, obtained by deleting $\mathbb{N}$ from $\mathbb{Z}$;
b) the collection B obtained combining the double of each negative number in $\mathbb{Z}$ and the collection of numbers from ① **- 8** to **6**①;
c) the collection D obtained by deleting from $\mathbb{N}$ its first and last items and then combining the resulting collection with B;
d) the collection E obtained combining A and B.

There are easily identifiable collections of which we are currently unable to determine the measure. Two of these are the collections of the odd and even numbers in $\mathbb{N}$. In order to ensure that they are measured, and to obtain a general method to extend the family of measured collections, we need to make some preliminary observations. To fix ideas, let us focus on the odd and even numbers in $\mathbb{N}$. The even numbers in $\mathbb{N}$ are characterised by the fact that their remainder when divided by two is zero. The odd numbers in $\mathbb{N}$, by contrast, all have remainder 1 when divided by two. We note that:

- division by two can only produce two remainders, 0 or 1;
- in the first case, it identifies an even number in $\mathbb{N}$;
- in the second case, it identifies an odd number in $\mathbb{N}$;
- $\mathbb{N}$ is split into the disjoint collections of odd and even numbers.

We call the collection of all items in $\mathbb{N}$ that have the same remainder when divided by two a **remainder class modulo** 2. We say that there are exactly two remainder classes modulo 2 and that thy subdivide $\mathbb{N}$ into two disjoint parts (no item occurs in both). The same situation arises if we consider a different divisor. For instance, there are three remainder classes modulo 3, which split $\mathbb{N}$ into three disjoint parts (they cannot share any item because the remainder of division by a number is uniquely determined). By the same clue, there are four remainder classes modulo 4 and n remainder classes modulo n, with n expressible in base ten. Since each choice of n determines a corresponding subdivision of $\mathbb{N}$ and the cells of the subdivision increase as n does, it seems convenient to introduce a postulate that assigns these subdivisions fractional measures of ①, which always sum up to ①.

For instance, we want a postulate that assigns measure ①$/2$ to each of the two remainder classes modulo 2, in such a way that $2($①$/2) =$ ①. In the same vein, the remainder classes modulo 3 will have measure ①$/3$ and $3($①$/3) = 1$. In general:

- **Postulate 3**: Each remainder class of $\mathbb{N}$ *modulo* n, with n expressible in base ten, has measure ①$/n$ and $n($①$/n) =$ ①.

Set $n = 2$. For any $m > 0$, with m expressible in base ten, $m + m = 2m$ is still expressible in base ten. By Postulate 1, $2m <$ ① holds. By Postulate 3, we must have:

$$\frac{①}{2} + \frac{①}{2} = 2\frac{①}{2} = ①,$$

whence $m <$ ①$/2$ for each m expressible in base ten. If this were not the case we would have ①$/2 \leq m$, from which we can deduce ① $=$ ①$/2 +$ ①$/2 \leq m + m = 2m$, contradicting Postulate 1. Thus, in particular, $0 <$ ①$/2$. The formal property to the effect that, if $0 < x$, then $x < x + x$, leads, courtesy of Postulate 2 (setting $x =$ ①$/2$), to the inequality ①$/2 <$ ①. Because $\mathbb{N}$ is completely counted by ① numerals, there follows that, at some point along the numeral tape used for the count, between 1 and ①, we are bound to encounter the numeral ①$/2$, which, by Postulate 3, must appear exactly half-way through the count. By Postulate 2, we know that ①$/2$ is to be preceded by ①$/2 - 1$ and followed by ①$/2 + 1$. This enables us to add more detail to a description of the numeral alignment that produces a completed count of $\mathbb{N}$. In light of Postulate 3, the alignment now looks like this:

$$1, 2, 3, 4, \ldots, \frac{①}{2} - 1,\ \frac{①}{2}, \frac{①}{2} + 1, \ldots, ① - 1,\ ①.$$

By Postulate 3, ①$/2$ even numbers are counted along the above alignment. We also note that ①, being twice the integer ①$/2$ (①$/2$ is in $\mathbb{N}$ and, thus, an integer), *is even*. As a result, we deduce that the list of numerals corresponding to the even numbers in $\mathbb{N}$ must consist of ①$/2$ items. Furthermore, its last item must be ①. The list in question is therefore:

$$2, 4, 6, \ldots, ① - 4, ① - 2, ①.$$

We are now in a position to compare a completed count of $\mathbb{N}$ with one of the even numbers in $\mathbb{N}$, as follows:

$$1 \quad 2 \quad 3 \quad 4 \quad \dots \quad \frac{①}{2}-1 \quad \frac{①}{2} \quad \frac{①}{2}+1 \quad \dots \quad ①-1 \quad ①.$$

$$2 \quad 4 \quad 6 \quad 8 \quad \dots \quad ①-2 \quad ①.$$

It is noteworthy that the beginnings of both counts are clearly identifiable in base ten, while their terminal stages are not. This is why we were forced to treat the collections given by these counts as equivalent when using $\aleph_0$. Our new numeral system enables us to treat the collections as distinct, since we can measure both collection and produce the distinct counts that measure them. The same is true of the odd numbers, when compared with $\mathbb{N}$.

Exercise 7. In the following exercises, we work with a new bit of notation. We call $\mathbb{N}_{k,n}$ the remainder class modulo n of $\mathbb{N}$ whose elements yield the remainder k (clearly $0 \le k \le n-1$). In this notation $\mathbb{N}_{0,2}$ is the collection of the even numbers in $\mathbb{N}$.

a) For which values of k,n is $\mathbb{N}_{k,n}$ the collection of odd numbers?
b) Compare a completed count of $\mathbb{N}$ and one of $\mathbb{N}_{0,5}$;
c) For which values of k,n is $\mathbb{N}_{k,n}$ the collection of multiples of three? What is the measure of this collection?
d) Compare a completed count of $\mathbb{N}$ and one of $\mathbb{N}_{1,3}$;
e) Compare a completed count of $\mathbb{N}$ and one of $\mathbb{N}_{2,5}$;
f) Compare a completed count of $\mathbb{N}$, one of $\mathbb{N}_{0,2}$ and one of $\mathbb{N}_{0,4}$.

2.5.1 Even and odd numbers

We have until this point restricted our attention to the even numbers in $\mathbb{N}$. It is legitimate to talk about even and odd numbers in $\mathscr{N}$ or $\mathscr{Z}$ as well. A number in $\mathscr{N}$ or $\mathscr{Z}$ is or is not a multiple (finite or infinite, positive or negative) of 2: in the former case, it is even, in the latter case, odd. Since ① is even Postulate 2 implies that both $① - 1$ and $① + 1$ are odd. The first number, however, is odd and in $\mathbb{N}$, whereas the second number is odd and not in $\mathbb{N}$. We turn, for a slightly less straightforward illustration, to $①/2$: is it even or odd? We claim that it is even. To verify the claim, we must exhibit x such that $2x$ equals $①/2$. Equivalently, we have to solve the equation $2x = ①/2$. To this end, we consider the collection $\mathbb{N}_{0,4}$ of multiples of 4. By Postulate 3, the last collection has measure ①/4. By Postulate 2 we can compute:

$$① = \left(\frac{①}{4}+\frac{①}{4}\right)+\left(\frac{①}{4}+\frac{①}{4}\right) = 2\left(\frac{①}{4}+\frac{①}{4}\right).$$

Since Postulate 3 guarantees $2(①/2) = ①$, we conclude:

$$2\left(\frac{①}{4}+\frac{①}{4}\right) = 2\frac{①}{2},$$

that is, by Postulate 2, which allows us to cancel the common factor 2:

$$\frac{①}{4}+\frac{①}{4} = \frac{①}{2}.$$

We have just verified that the solution to $2x = ①/2$ is $x = ①/4$. This suffices to deduce that $①/2$ is even. It is also in $\mathbb{N}$. As a consequence, both $①/2-1$ and $①/2+1$ are odd numbers in $\mathbb{N}$.

Example 5. *Determine whether* $(①-5)①$ *is even or odd and whether it is in* $\mathbb{N}$. We know that ① is even and that $①-5$ is odd, since the latter precedes the even number $①-4$, which is clearly even (justify this remark). By Postulate 2, the product of an even and and odd number is even. It follows that $(①-5)①$ is even. In order to determine whether the last number is in $\mathbb{N}$, we consider the equivalent form $①^2-5①$. By Postulate 2, we can use the formal property:

$$\text{if } 0 < x < y \text{ then } x \cdot y < y \cdot y = y^2,$$

on the values $x = 6$ e $y = ①$ to obtain $6① < ①^2$. Subtracting $5①$ from both sides of the last inequality (which is allowed by Postulate 2), we reach the inequality $① < ①^2-5①$, which shows that $(①-5)①$ cannot be in $\mathbb{N}$.

Exercise 8.

a) Is $① \left(\frac{①}{2}\right)$ even or odd? Is it in $\mathbb{N}$?

b) Is $\frac{①}{7}$ even or odd? Is it in $\mathbb{N}$?

Exercise 9. Determine whether each of the following numbers is even or odd and whether it is in $\mathbb{N}$:

a) the number of items in $\mathbb{N}_{0,1}$;

b) $\frac{①}{5}$;

c) $\frac{①}{3} + 1$;

d) $① - \frac{①}{6}$;

e) $\frac{①}{3} + \frac{①}{4} - 2$;

f) $3\left(3 + \frac{①}{6}\right)$.

g) $\left(\frac{①}{5} - \frac{①}{7}\right) + \frac{①}{2}$;

h) $-\left(\frac{①}{3} + 1\right)\left(\frac{①}{3} - 1\right)$;

i) $\left(\frac{①}{2} - 3\right)\left(\frac{①}{7} - 3\right)(① - 1)$.

2.6 Fractions

Our transition from $\mathscr{N}$ to the broader environment $\mathscr{Z}$ is helpful, but suffers from certain limitations that it is advisable to overcome. These limitations are best understood in terms of what we can already do with the fractional measures introduced by Postulate 3. Suppose that we want to compute $①/2 - ①/3$. This is the difference between two integers, which we rightly expect to be an integer in $\mathscr{N}$, since $①/2 > ①/3$. We have a way of computing this difference, but it is not wholly satisfactory because it requires the direct consideration of collections. In an applied arithmetical calculus we wish to bypass such direct consideration because we wish to anticipate the results of counting in computation. It is however instructive to see how the direct consideration works, because it provides a clearer picture of what we want to achieve with a further extension of the numerical environment beyond $\mathscr{Z}$. The obvious way to evaluate $①/2 - ①/3$ is to consider a specific collection of $①/2$ items, delete from it $①/3$ items and then count the remaining items. To this end, we start from the collection of even numbers in $\mathbb{N}$, which has measure ①$/2$, and we subdivide it into the following three remainder classes:

$$\begin{array}{lcl} \mathbb{N}_{2,6} & = & \{2 \quad 8 \quad 14 \quad 20 \quad \dots \quad ①-10 \quad ①-4\} \\ \mathbb{N}_{4,6} & = & \{4 \quad 10 \quad 16 \quad 22 \quad \dots \quad ①-8 \quad ①-2\} \\ \mathbb{N}_{0,6} & = & \{6 \quad 12 \quad 18 \quad 24 \quad \dots \quad ①-6 \quad ①\}. \end{array}$$

Moving from the top to the bottom row and from left to right, we can verify at a glance that the above subdivision does list all even numbers in $\mathbb{N}$. By Postulate 3 tells us that each of the three collections into which we have split the even numbers has measure ①$/6$. Using Postulates 2 and 3, we can also verify that $2($①$/6) =$ ①$/3$ (supply details for the last claim). As a consequence, once we delete from $\mathbb{N}_{0,2}$, a collection of measure ①$/2$, the parts $\mathbb{N}_{2,6}$ and $\mathbb{N}_{4,6}$, whose combined measure is ①$/3$, we obtain a collection of measure ①$/6$. We have verified, with respect to a specific instance:

$$\frac{①}{2} - \frac{①}{3} = \frac{①}{6}.$$

It is not difficult to realise that, following a similar strategy, the evaluation of ①$/2 +$ ①$/3$ is also within reach. We again use the three remainder classes employed to evaluate the difference ①$/2 -$ ①$/3$. These remainder classes modulo 6 do not include $\mathbb{N}_{1,6}$ and $\mathbb{N}_{3,6}$. The latter two classes, when combined, give rise to a collection of measure ①$/3$. Because of this, we can illustrate the sum ①$/2+$①$/3$ by means of the combination of $\mathbb{N}_{0,2}$, $\mathbb{N}_{1,6}$ and $\mathbb{N}_{3,6}$. Since the collection $\mathbb{N}_0$ can be split into three collections of measure ①$/6$ each, we are in effect combining five collections of measure ①$/6$. Postulate 2 implies that:

$$\frac{①}{2} + \frac{①}{3} = 5\frac{①}{6}.$$

It is of the essence to pause and note three things. First, in our last calculations we have presupposed a definition of arithmetical sum as the measure of a combination of disjoint collections. We take applied arithmetical sums of integers to be precisely this. Second, even if we are entitled to write the integer $5($①$/6)$, it would be convenient, but currently impossible, to assert its equivalence with ①$(5/6)$: this we cannot do because we have not introduced fractions and thus cannot express the integer $5($①$/6)$ as the product of a fraction and the integer ①. Third, the detour on specific collections for the sake of evaluating ①$/2 -$ ①$/3$ provided a way to work with a common denominator, namely 6 for the terms involved in the difference. The last two remarks show that it would be natural and helpful to extend our arithmetical capabilities to fractions in a natural way, in order to identify intuitively equivalent numerical terms and in order to replace the consideration of specific partitions of collections into remainder classes with a direct computation of common denominators. Before doing this, though, it is instructive to do some practice with evaluating sums and differences of fractional measures by considering specific collections.

Exercise 10. Use the strategy just described to evaluate the following sums and differences in $\mathscr{Z}$:

a) $\frac{①}{6}+\frac{①}{2}$, $\frac{①}{6}+\frac{①}{3}$, $\frac{①}{6}-\frac{①}{2}$;

b) $\frac{①}{2}-\frac{①}{4}$, $\frac{①}{2}+\frac{①}{4}$, $\frac{①}{4}-\frac{①}{2}$;

c) $\frac{①}{3}-\frac{①}{4}$, $\frac{①}{4}-\frac{①}{3}$, $\frac{①}{3}+\frac{①}{4}$.

We now turn to an environment in which all items have the form a/b, with a,b in $\mathscr{Z}$ and $b \neq 0$. We call the new environment $\mathscr{Q}$. Some of its items are:

$$\frac{15}{16}, \frac{2①}{7}, \frac{18}{①^2}, \frac{①^2}{5①^4}.$$

There is no unique way of designating the same item, since e.g. $2/2 = 1 = ①/①$ and $2/4 = 1/2 = 3①^2/6①^2$. Even if multiple representation of the same item are possible, we shall informally identify them. We now extend Postulates 1, 2, 3 to $\mathscr{Q}$[4]. By the new version of Postulate 2, every $x \neq 0$ in $\mathscr{Q}$ has a multiplicative inverse. For instance, 2 has multiplicative inverse $1/2$. As for ①, its multiplicative inverse is $1/①$ and:

$$① \cdot \frac{1}{①} = \frac{①}{①} = 1.$$

By Postulates 1 and 2, $n < ①$ and the formal property:

$$\text{if } 0 < x < y \text{ then } \frac{1}{x} > \frac{1}{y}.$$

entail $1/① < 1/n$, for each $n \neq 0$ expressible in base ten. In particular:

[4]The reformulation of Postulate 2 requires a reference to $\mathbb{Q}$ (the set of rational numbers, discussed below), as opposed to $\mathbb{N}$

$$\frac{1}{①} < \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots.$$

We call $1/①$ a positive **infinitesimal**.

Definition: a positive number smaller than $1/n$, for each n expressible in base ten, is a *positive infinitesimal*.

Since $1/n = 2/2n$, we can deduce:

$$\frac{2}{①} < \frac{2}{4} = \frac{1}{2}, \frac{2}{6} = \frac{1}{3}, \frac{2}{8} = \frac{1}{4}, \dots,$$

and find that $2/①$, the multiplicative inverse of $①/2$, the measure of $\mathbb{N}_{0,2}$, is a positive infinitesimal. A negative infinitesimal is defined in the obvious way.

Exercise 11. Verify the following inequalities, in which n is expressible in base ten:
(a) $\frac{1}{①^2} < \frac{1}{n①}$; $(b)\frac{1}{①^3} < \frac{1}{n①^2}$, $(c)\frac{1}{①^{①}} < \frac{1}{n①^n}$.

There is an infinitely large number of distinct infinitesimals. For instance, each $1/①^k$, with $k \in \mathbb{N}$, is an infinitesimal. Thus, there are at least ① distinct infinitesimals. There also are infinitesimals smaller than an infinitely large number of other infinitesimals. For instance, the infinitesimal $1/①^2$ is smaller than $①^2 - (① + 1)$ other infinitesimals of the form $1/a$, with a in $\mathcal{N}$ (note that $[①^2 - (① + 1)] - ① = (① - 1)^2 > 0$, i.e. $①^2 - (① + 1) > ①$).

Exercise 12. Draw a table in which all values of a in $\mathcal{N}$ for which the inequality $1/①^2 < 1/a$ holds are listed. *Suggestion*: use a table in which rows have exactly ① slots. The last row contains $① - 1$ elements.

We define a rational number (up to equivalence) as a number of the form m/n, where m, n are $\mathbb{Z}$ and $n \neq 0$. The collection of rational numbers is $\mathbb{Q}$. Our environment $\mathcal{Q}$ includes $\mathbb{Q}$. There is no easy way of evaluating the size of $\mathbb{Q}$, though, precisely on account of the fact that equivalent forms like $2/3$ and $4/6$ count for one item. It is, on the other hand, easy to determine whether an item from $\mathcal{Q}$ is or not in $\mathbb{Q}$. For instance, since $1, -5, ①$ are nonzero and in $\mathbb{N}$, the fractions $-\frac{1}{5}$ and $1/①$ are both in $\mathbb{Q}$. By contrast, since $5①$ is in $\mathcal{N}$ but not in $\mathbb{N}$, the fraction $1/5①$ is a positive infinitesimal of $\mathscr{Q}$ not in $\mathbb{Q}$. In a similar fashion, for any n expressible in base ten, $1/①^n < 1/①$ is an infinitesimal not in $\mathbb{Q}$.

Exercise 13. Determine which ones among the following terms designate items in $\mathbb{Q}$:
(a) $\dfrac{①-1}{①}$; (b) $1-\dfrac{1}{①^2}$; (c) $\dfrac{1}{①}+\dfrac{2}{①-2}$; (d) $\dfrac{7①}{3}$; (e) $\dfrac{0}{2①}$.

We conclude by observing that, in view of the version of Postulate 2 relative to the environment $\mathscr{Q}$, the familiar rules to sum and multiply fractions continue to apply.

Exercise 14. Simplify the following expressions:

a) $\dfrac{1}{7}\left(\dfrac{①^2}{2}+\dfrac{①^2}{3}-\dfrac{①^2}{4}\right)$;

b) $\left[\left(\dfrac{①}{4}+\dfrac{1-①}{2①}\right)-\dfrac{①-2}{4}\right]$;

c) $\left(\dfrac{①}{3}-\dfrac{①}{4}\right)\left(\dfrac{3}{①}+6+3①\right)$;

d) $\left[\left(\dfrac{1}{①}+\dfrac{1}{①^2}-\dfrac{1-①^2}{①^3}\right)-\dfrac{3}{①}\left(\dfrac{1}{①}-\dfrac{3+①}{3①}\right)\right]$;

e) $-\dfrac{1}{3}\left[\left(\dfrac{4①^2-3①-1}{①^2}\right)\left(\dfrac{①}{①-1}-\dfrac{4①}{4①+1}\right)\right]$.

2.7 Numeral representation

In this chapter we have discussed a new numeral system, due to Yaroslav Segeyev and we saw how it can be used to develop an applied arithmetical calculus that involves both finite and infinite collections and allows computations involving finite, infinitely large and infinitesimal, signed quantities. We have built this calculus in stages, beginning with the extended environment $\mathscr{N}$ and moving to $\mathscr{Z}$ and $\mathscr{Q}$. It seems appropriate to point out, if only in passing, that the items in $\mathscr{Q}$ have records in base ①, much like those in $\mathscr{N}$ or $\mathscr{Z}$. In these two environments we never appeal to negative powers of ①, but we do in order to obtain numeral records. For instance, using the notational conventions from 2.3.1, we can assign to the numerical term:

$$①^4 + \frac{3①}{4} + 7 + \frac{2}{①^2}$$

the record:

$$1 \cdot ①^4 0.75 \cdot ①^1 7 \cdot ①^0 2 \cdot ①^{-2}.$$

Exercise 15. Provide records in base ① for each of the following terms: (a) $\frac{1}{①}$; (b) $1 - \frac{1}{①^2}$; (c) $\frac{3①^2}{①}$; (d) $\frac{① - 1}{2①}$.

The reader familiar with the standard presentation of numerical sets may be surprised to encounter environments like $\mathscr{N}, \mathscr{Z}, \mathscr{Q}$, in which Postulates 1, 2, and 3 hold. From a logical point of view, the introduction of what is in essence our $\mathscr{N}$ is presented in axiomatic terms in [**15**]. Extensions to integers and rationals are possible in the axiomatic framework developed by [**18**]. An alternative route to obtaining them, which depends on Lolli's results, is sketched in [**24**].

3. Sequences and Series

3.1 Infinite Sequences

The arithmetical calculus of measures presented in chapter 2 has one clear advantage over traditional arithmetic. It can deal with completed counts of collections like $\mathbb{N}, \mathbb{N}_{0,2}$ or $\mathbb{Z}$, which can be presented as infinitely long sequences of items. From our point of view, such sequences are a natural object of study. Naturalness amounts in the first instance to the satisfactory avoidance of complications that arise when infinite sequences are studied by standard means. The reliance over a numeral system like the one in base ten and the subsequent development of an arithmetic of finite quantities lead to an array of basic techniques not suited to manage infinite collections. It is as a result of this shortcoming that an expansion of the basic concepts and techniques occurs, which standardly hinges on the introduction of the concept of limit.

We are in a position to pursue an alternative path because we have decided not to be content with an arithmetical calculus computationally inadequate to handle infinite collections, and which must call for supplementary aid in order to cope with them, but instead to go through the effort to extend our arithmetical calculus to a point where it is already well equipped to deal with infinite processes like sequences. For this very reason, we can provide a non-trivial study of sequences independent of the limit operation. Our work on infinite sequences in the first part of this chapter will also provide the platform to develop the study of infinite series, with which the chapter ends.

Our starting point to describe sequences will not be the standard notion of convergence but the arithmetical notion of a **complete sequence**. When we work with base ten numerals, a generic infinite sequence is usually described as follows:

$$a_1, a_2, a_3, \dots, a_n, a_{n+1}, \dots$$

It is presupposed that the indices of the a_i exhaust $\mathbb{N}$. Since, however, no measure for a sequence can be provided, the deletion of all items with an odd index leads to the sequence:

$$a_2, a_4, a_6, \dots, a_{2n}, a_{2n+2}, \dots$$

which, as far as its length is concerned, looks equivalent to the original sequence (recall that even indices and the indices in $\mathbb{N}$ determine collections to which we classically assign the same measure of size $\aleph_0$). In the especially simple case in which $a_1 = a_2 = a_3 = a_4 = \dots$, both sequences would appear completely indistinguishable. The principle of Euclidean Size, which we have endorsed, requires that they should be distinct: once we work with numeral terms involving ①, we can turn their distinctness into a numerical discrimination. To make this apparent, we adopt the following:

Definition an infinite sequence is complete if, and only if, the collection of its elements has measure ①.

We also refer to the measure of the collection of elements in a sequence as its length. From now on, we shall use the terms length and measure interchangeably. In particular, the above definition of a complete sequence may be rephrased by saying that its length is ①. Clearly, the sequence:

$$1, 2, 3, \dots, \frac{①}{2} - 1, \frac{①}{2}, \frac{①}{2} + 1, \dots, ① - 2, ① - 1, ①$$

is complete. A generic, complete sequence will be described as follows:

$$a_1, a_2, a_3, \dots, a_{①-2}, a_{①-1}, a_{①}.$$

Clearly a_1 is the first item in the sequence and $a_{①}$ the last. We may also adopt an alternative description, which will prove useful, and simply specifies on which numerical collection the indices of the terms in the sequence vary. In the case of the above complete sequence we have:

$$a_i = i \quad \text{e} \quad i = 1, 2, 3, \dots, ① - 2, ① - 1, ①.$$

The presentation of a sequence just introduced enables us to determine an infinite sequence simply by providing the form of its generic term a_i and a numerical measure of its length. For instance, a specific complete sequence is:

$$a_i = 2i \ ; \ i = 1, 2, 3, \ldots, ① - 2, ① - 1, ①.$$

In more explicit notation, the same sequence is:

$$2, 4, 6, \ldots, 2① - 4, 2① - 2, 2①.$$

A shorter sequence is given by:

$$a_i = 2i \ ; \ \ i = 1, 2, 3, \ldots, ①/2 - 1, ①/2,$$

which, in more explicit notation, is:

$$2, 4, 6, \ldots, ① - 2, ①,$$

namely the sequence of even numbers in $\mathbb{N}$, which is not complete.

Exercise 16. For each of the following infinite sequences, write its first and last three elements and decide whether it is complete or not:

a) $a_i = 2i - 1 \ ; \ i = 1, 2, \ldots, ①.$

b) $\mathbb{N}_{1,2}.$

c) $a_i = ① - i \ ; \ i = 0, 1, 2, \ldots, ① - 1.$

d) $a_i = \frac{1}{i^2} \ ; \ i = \frac{①}{2}, \frac{①}{2} + 1, \ldots, 3\frac{①}{2}.$

e) $a_i = \frac{3i}{4} \ ; \ i = -\frac{①}{3}, -\frac{①}{3} + 1, \ldots, \frac{2①}{3} - 1, \frac{2①}{3}.$

There are two ways in which a sequence may fail to be complete: it may be, respectively, longer or shorter than a complete sequence. In the first case we talk about an *extended sequence*. By way of illustration, the sequence:

$$0, 1, 2, 3, \ldots, ① - 1, ①, ① + 1, \ldots, 2① - 1, 2①$$

is extended and of length $2① + 1$, the same as the measure of $\mathbb{Z}$.

Exercise 17. For each of the following sequences, write the first, middle and last three terms. Find out which ones are complete sequences and which ones are extended sequences.

a) $\mathbb{N}_{1,2}$ followed by $\mathbb{N}_{1,3}$ and by $\mathbb{N}_{2,3}$.

b) $\mathbb{N}_{0,6}$ preceded by $\mathbb{N}_{0,2}$ and followed by the sequence $a_i = ①i\ ;\ i = 3,4,\ldots,\frac{①}{2}$.

c) $a_i = ① - i\ ;\ i = 0,1,2,\ldots,①$.

d) $a_i = 1/2^i\ ;\ i = -①, -① + 1, \ldots, ① - 2, ① - 1$.

e) $a_i = 3i\ ;\ i = -\frac{①}{3}, -\frac{①}{3} + 1, \ldots, \frac{①}{3}$.

We offer at this point some brief remarks addressed to the leader familiar with sequential convergence. As we have already pointed out, sequences whose convergence is to be studied are standardly described in the following manner:

$$a_1, a_2, a_3, \ldots,$$

where the indices take all values in $\mathbb{N}$. In our terminology, only complete sequences are taken into account, even though their completion at the ①-th term cannot be represented. As a consequence, convergence provides, from our point of view, only an estimate of the values attained upon completion. For instance the two infinite sequences below:

$$a_i = \frac{1}{i}\ ;\ i = 1,2,3,\ldots \text{ e } b_i = \frac{1}{i^2}\ ;\ i = 1,2,3,\ldots$$

both converge to 0 as n grows larger and larger (in symbols $n \to \infty$). When, however, numerical specification of completions are available, the limit 0 appears to be a rough estimate of distinct infinitesimal terminations. In particular, we have $a_① = 1/①$ e $b_① = 1/①^2$. Even though $a_①, b_①$ are both positive infinitesimals, the second is strictly smaller than the first. Substituting the limit operation for direct numerical evaluations, we lose a certain amount of numerical accuracy. Another illustration of the same phenomenon is provided by the sequences:

$$a_i = i \; ; \; i = 1,2,3,\ldots \text{ and } b_i = i^2 \; ; \; i = 1,2,3,\ldots.$$

Each diverges. In standard notation, one may express this fact as an identify of behaviour:

$$\lim_{i \to \infty} a_i = \infty = \lim_{i \to \infty} b_i.$$

As in the case of the numeral store used in Chapter 1, ∞ provides a relatively inaccurate evaluation. It identifies completions that can be numerically discriminated in base ①. We see that:

$$a_{①} = ① < ①^2 = b_{①}.$$

The complete sequences share a feature captured by the evaluation in terms of convergence, namely both grow beyond any fixed quantity expressible in base ten. However, they attain distinct values, whose difference is in particular infinitely large. Divergence does not take into account that the sequences may achieve widely different completions.

Exercise 18.

a) Consider the complete sequence $a_i = \frac{i}{2}$; $i = 1,2,\ldots,① - 1,①$. How many of the even numbers in $\mathbb{N}$ does it contain? How many of the odd numbers in $\mathbb{N}$?

b) Describe a complete sequence that contains no infinitesimals, ends with 0 and only involves whole numbers.

c) A complete sequence is decreasing if it satisfies the inequalities: $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_{①-1} \geq a_{①}$. Describe a decreasing sequence whose terms are all infinitesimal and such that $a_{①} = 0$.

d) Describe two complete sequences whose last terms $a_{①}$ and $b_{①}$ are infinitely large and have an infinitely small difference.

e) Determine the last term of: $a_i = (-1)^i$; $i = 1,2,\ldots,① - 1,①$ e di $a_i = (-1)^i$; $i = 1,2,3,\ldots,①/3 - 1$.

3.1.1 Sequences of indexed place-holders

The next rewarding task to turn to is developing techniques to sum the terms of an infinite sequence. Before turning to it, though, it is of some interest to look at infinite sequences from a basic combinatorial perspectives. We no longer want to take terms as given, but rather work with a complete sequence of indexed place-holders, which we can later fill in with numbers. Before being filled, the place-holders may look as follows:

$$\Box_1, \Box_2, \Box_3, \Box_4, \Box_5, \Box_6, \dots \Box_{\textcircled{1}-4}, \Box_{\textcircled{1}-3}, \Box_{\textcircled{1}-2}, \Box_{\textcircled{1}-1}, \Box_{\textcircled{1}}.$$

To dwell on this setup a bit will help us realise a crucial difference between studying infinite sequences in, say, base ten, and studying them with our supplementary arithmetical resources. In the former case, it may look as if we are always able to replicate one operation of insertion into an empty slot, without ever running out of available slots. In the latter case, it becomes clear that certain infinitely repeated operations may require so many stages that there are not enough empty slots to carry them all out. To appreciate the last remarks, consider the standard description of an infinite sequence of indexed place-holders:

$$\Box_1, \Box_2, \Box_3, \Box_4, \Box_5 \Box_6, \dots$$

and to have a supply of terms:

$$a_1, a_2, a_3, a_4, a_5, a_6, \dots$$

that can be inserted into prescribed empty places. We might for instance choose to single out the place-holders with index a multiple of 4 and decide that the a_i with an even index should occupy them. Secondly, we use the a_i of odd index to occupy the remaining, empty places with an even index. The result of the operations just described is:

$$\Box_1, a_1, \Box_3, a_2, \Box_5, a_3, \dots .$$

If we replace each place-holder by the corresponding a_i, we obtain from the sequence of the a_i the empty sequence of place-holders. From this point of view, they look equivalent. However, the operations we have just carried out seem to suggest that we should regard the sequence of the a_i as a part of the sequence of indexed place-holders. Without further numerical resources, it is unclear which one of these situation is suggestive of a determinate comparison. In other words, it is unclear whether the a_i should determine a sequence that is longer or shorter than another sequence. One standard way out of this predicament is to identify all lengths of sequential arrangements indexed in base ten. For reasons that we have already enlarged upon, this strategy, however viable, forces computational constraints that we wish to escape. It is for this reason that we prefer to tackle the last problem in an alternative manner and introduce a numerical description of sequential completions. If the a_i are taken to be a completed sequence, and so are the indexed place-holders, then moving the ①/2 items a_i of an even index into the place-holders indexed by a multiple of 4 is no longer possible after we position the first ①/4 items, namely:

$$a_2, a_4, \ldots, a_{①/2}.$$

Having positioned them, we occupy infinitely many place-holders as follows:

$$\Box_1, \Box_2, \Box_3, a_2, \Box_5, \Box_6, \ldots a_{\frac{①}{2}-1}, \Box_{①-3}, \Box_{①-2}, \Box_{①-1}, a_{\frac{①}{2}}.$$

The remaining place-holdrs with an even index are only ①/4, since we have used up ①/4 of them already. They can only fit the first ①/4 a_i with an odd index, namely:

$$a_1, a_3, a_5, a_7, \ldots, a_{\frac{①}{2}-3}, a_{\frac{①}{2}-1}.$$

When we move the last items into the respective positions, we obtain:

$$\Box_1, a_1, \Box_3, a_2, \Box_5, a_3, \ldots, \Box_{①-3}, a_{\frac{①}{2}-1}, \Box_{①-1}, a_{\frac{①}{2}}.$$

Far from shrinking the original, complete sequence of the a_i to half its length, as the standard, base ten description seemed to suggest, we have only used half of our supply of even-indexed and odd-indexed terms. Since these terms determine a collection of measure ①/2, it is no wonder that they cannot fill all of the initially given indexed place-holders, since there are twice as many of them. Whereas it is meaningful to say that, given an infinite sequence (of any length), the replacement of each one of its terms by an empty place-holder leads to an infinite sequence of place holders of the same length, more complicated operations that depend on more accurate specifications of length may produce equivalences where stronger numerical resources are able to recover differences.

Exercise 19. Consider a complete sequence of indexed place-holders. Sequentially insert the elements of $\mathbb{N}_{0,3}$ in the positions of even index. Then, sequentially insert the elements of $\mathbb{N}_{1,3}$ in the positions of odd index, starting from the index $①/4-1$. Describe the outcome of these operations and identify the infinitely long blocks of consecutive, empty place-holders unaffected by the operations performed. Morevoer, specify the length of each one of these blocks.

3.2 Infinite Series

Given an infinite sequence of specifiable length, the easiest ways of operating upon it arithmetically is to compute the sum of its terms. Thus, given for instance the complete sequence:

$$a_1, a_2, a_3, \ldots, a_{\text{①}-1}, a_{\text{①}},$$

we wish to evaluate the infinitely long summation:

$$a_1 + a_2 + a_3 + \ldots + a_{\text{①}-1} + a_{\text{①}}.$$

We call the last expression *the infinite series associated with the sequence* $a_1, \ldots a_{\text{①}}$. We call its numerical value the *sum of the series*. As the last terminological choice shows, we usually drop the adjective 'infinite' from the locution 'infinite series' and simply speak of series. In the remainder of this chapter we shall refer to series using both the explicit notation for a summation and the 'sigma' notation:

$$\sum_{i=1}^{k} a_i = a_1 + a_2 + \ldots + a_{k-1} + a_k$$

where the symbol 'Σ', i.e. the capitalised Greek letter 'sigma', designates a summation of k consecutive terms, namely $a_1, \ldots, a_k$. Here k stands in general for a number in $\mathscr{L}$.

There is an obvious correspondence between infinite sequences and series. The summands of a series determine the terms of a sequence and viceversa. If a sequence is complete, the corresponding series is a *complete series*. An extended sequence gives rise to an *extended series*. Finally, we make use of infinite, constant sequences of a specified length to determine corresponding *constant series*. An infinite sequence is constant when all of its terms are equal. The complete sequence:

$$\underbrace{a, a, a, a, \ldots, a, a}_{\text{① } times}$$

for a specifiable choice of a, is a constant sequence that determines the constant series:

$$\underbrace{a + a + a + a + \ldots + a + a}_{\text{① } times},$$

whose sum is ①a. When a sequence (complete or otherwise) is constant, we refer to any one of its terms as its *coefficient*.

Exercise 20.

a) Find the sums of the complete, constant series of coefficients $3, 4, 10$ respectively. Determine the positive differences between these sums.

b) Given the constant sequence of coefficient 2 and length 15①, find the sum of the corresponding infinite series.

c) Given the complete, constant sequence of coefficient 1, let s be the corresponding series. Determine a complete series t such that $t + s = -1$.

d) Given a constant sequence of coefficient ①/2 and length ①/3, determine the sum of the corresponding series.

We remarked that, in base ten, constant sequences with the same coefficient all look identical. The coefficient a uniquely determines the sequence:

$$a, a, a, a, \ldots$$

whose corresponding series has no sum. In standard notation, one also says that the sum of the series is ∞ or $-\infty$, depending on the sign of a. This conclusion is insensitive to the choice of a. This situation of complete uniformity gives way to a host of numerical discriminations in base ①. It is possible, for instance, to describe two constant sequences with the same coefficients and such that one is an extension of the other, or possibly a complete extension of the other. If coefficients differ, then the sums of constant series of coefficients expressible in base ten can be computed, once the number of summands is given, and the resulting numerical values may be subjected to further numerical calculations that never collapse values (as in the case of the term $\infty + \infty$) or produce indeterminate form (as in the case of the term $\infty - \infty$). For instance, given the coefficients $3, 7$, we may only consider the constant sequences:

$$3, 3, 3, \ldots \quad \text{e} \quad 7, 7, 7, \ldots.$$

The corresponding series diverge to ∞. Their sums, if evaluated at ∞, produce an indeterminate difference and a sum equal to each of them. When we move to the numeral system in base ①, we confront a very different situation. Among other things, we can state and solve problems that could not be formulated in standard terms.

Example 6. Specify the length of an infinite, constant sequence s of coefficient 3 and the length of an infinite, constant sequence t of coefficient 7 in such a way that: (a) the sum of the series associated with s be smaller than the sum of the series associated with t; (b) the sum of the series associated with s be greater than the sum of the series associated with t; (c) the sum of the series associated with s be equal to the sum of the series associated with t.

Solution: (a) we may choose two complete sequences, each of length ①. Let us call $S(s), S(t)$ the sums of the series associated to the sequences s, t respectively. We have $S(s) = 3① < 7① = S(t)$; (b) we let s have length 3① and keep t complete. Then $S(s) = 9① > 7① = S(t)$; (c) we let s have length ①/6 and t have length ①/14.

Exercise 21. Specify the lengths of an infinite sequence s of coefficient -2 and of an infinite sequence t of coefficient 1 in such a way that:

a) $S(s) + S(t) = 0$;

b) $S(s) + S(t) < 0$;

c) $S(s) + S(t) > 0$.

The calculations carried out up to this point are very simple. It is noteworthy that even basic arithmetical calculations involving infinity are without the computational reach of standard arithmetic.

3.2.1 Arithmetical Series

A completed count of $\mathbb{N}$ determines the complete sequence:

$$1, 2, 3, 4, \ldots, ① - 1, ①.$$

We wish to find the sum of the corresponding series, that is:

$$\sum_{i=1}^{①} i = 1 + 2 + 3 + 4 + \ldots + ① - 1 + ①.$$

We do so exploiting a trick attributed to Carl Friedrich Gauss (1777–1855). We arrange the terms of the sequence along two rows, thus:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & \dots & ①; \\ ① & ①-1 & ①-2 & ①-3 & \dots & 1. \end{array}$$

It is easy to find the sum of the 2① arrayed in two rows: it suffices to note that each column of the array has sum $① + 1$ and that there are ① columns. The sum of all terms is $①(① + 1)$. On the other hand, the sume we are interested in only involves ①. However, because each term from the first row occurs exactly once in the second row, the sum we are looking for must be half of $①(① + 1)$. In symbols:

$$\sum_{i=1}^{①} i = 1 + 2 + \ldots + (① - 1) + ① = \frac{①(① + 1)}{2}.$$

Exercise 22.

a) Compute $1 + 2 + 3 + \ldots + 34 + 35$;

b) Compute $1 + 2 + 3 + \ldots + \left(\frac{①}{2} - 1\right) + \frac{①}{2}$;

c) Compute $1 + 2 + 3 + \ldots + \left(\frac{①}{3} - 2\right) + \left(\frac{①}{3} - 1\right)$.

d) Compute the sum of the complete series of positive, even numbers: $2 + 4 + 6 + \ldots + 2① - 2 + 2①$ (*Suggestion*: write each term as a sum of two equal terms).

e) Use Gauss' trick to find the sum of the even numbers in $\mathbb{N}$.

f) Find the sum of the odd numbers in $\mathbb{N}$.

Since we now know the sum of $1 + 2 + \ldots + (① - 1) + ①$, we can exploit this knowledge to find the sums of other series. Our focus will be complete **arithmetical progressions**. A complete arithmetical progression is a complete sequence whose terms are generated by repeatedly adding a fixed increment to an initial term. If the initial term is a and the increment d, the complete arithmetical progression starting with a and of increment d is:

$$a, a + d, a + 2d, \ldots, a + (① - 2)d, a + (① - 1)d.$$

We use the shorthand $\mathsf{A}_{a,d}$ for the last sequence. The corresponding series is:

$$\sum_{i=0}^{①-1} a+id =$$
$$a+(a+d)+(a+2d)+(a+3d)+\ldots+(a+(①-2)d)+(a+(①-1)d).$$

We take for granted (a justification for this claim will be offered at the end of this chapter) that the sum of a series whose length is specifiable is not affected by a rearrangement of its terms. This implies, in particular, that we can rewrite the last arithmetical series as follows:

$$(\underbrace{a+a+a+\ldots a}_{①times})+d(1+2+3+\ldots+(①-1)) = ①a+\frac{d①}{2}(①-1).$$

Exercise 23. Determine the sums of the series associated with the following complete arithmetical progressions: $\mathsf{A}_{1,2}, \mathsf{A}_{2,2}, \mathsf{A}_{1,3}, \mathsf{A}_{2,3}, \mathsf{A}_{3,3}, \mathsf{A}_{①,4}, \mathsf{A}_{2,①/3}$.

If an arithmetical progression is not complete, the sum of the corresponding series can still be computed. Let λ be the progression's length. The sum of its terms can be written as:

$$a\lambda + d(1+2+3+\ldots+(\lambda-1)).$$

Using Gauss' trick again, we find that:

$$1+2+3+\ldots+\lambda = \frac{\lambda(\lambda+1)}{2}.$$

Exercise 24.

a) Justify the last equality;

b) Determine the sum of the series associated with $\mathbb{N}_{0,4}$;

c) Determine the sum of the series associated with $\mathbb{N}_{1,5}$.

So far we have restricted attention to positive terms. The restriction is inessential. An interesting case is that of alternating signs. An alternating sequence is one in which any two consecutive terms have opposite sign. The series associated with an alternating sequence is its *alternating series*. We begin with examining complete, alternating series. The simplest case is one where we have constancy, up to a sign, of the form:

$$\underbrace{a, -a, a, \ldots, -a, a, -a}_{\text{①}\ times}.$$

We now compute the sum of the corresponding alternating series. We may assume $a > 0$ (analogous considerations apply when $0 > a$). Because we know that there are ①$/2$ positive terms and ①$/2$ negative ones, we sum the terms with the same sign separately first. Positive terms sum up to a①$/2$, negative ones to $-a$①$/2$. The alternating series has sum 0. A different way of reaching the same conclusion consists in associating pairs of terms and exploiting Postulate 3, more precisely the fact that ① $= 2($①$/2)$. Then the alternating series looks like this:

$$\underbrace{(a-a)+(a-a)+\ldots+(a-a)}_{\text{①}/2\ volte}$$

and, because $(a-a) = 0$, it is in effect nothing but a sum of ①$/2$ terms or equal to zero.

Exercise 25. The complete sequence starting with 1 and alternating 1 with -1 determines *Grandi's series* $\sum_{i=0}^{\text{①}-1}(-1)^i$ (after Luigi Guido Grandi (1671–1742)).

a) Find the sum of Grandi's series.
b) Which extensions of Grandi's series have the sum of Grandi's series? Which extensions have a different sum?
c) Find the sum of the first $\frac{2\text{①}}{3} - 7$ terms of Grandi's series.

We call $\mathsf{A}'_{a,d}$ the alternating arithmetical progression obtained from $\mathsf{A}_{a,d}$ by multplying the second, fourth, $\ldots, 2n$-th term ($n = 1, \ldots,$ ①$/2$) times -1. For instance, $\mathsf{A}'_{1,1}$ is $1, -2, 3, -4, \ldots,$ ① $-1, -$①. In full analogy with the approach adopted for arithmetical progression, we may rely on $\mathsf{A}'_{1,1}$ in order to determine the sum of an alternating series corresponding to a sequence obtained by priming some arithmetical progression $\mathsf{A}_{a,d}$. To see why the approach works, note that the series associated with $\mathsf{A}'_{a,d}$ is:

$$a + (-a-d) + (a+2d) + (-a-3d) + \ldots + (-a - (\text{①} - 1)d),$$

which, suitably rearranged, yields:

$$\underbrace{a-a+a-a+\dots a-a}_{① \text{ volte}} - d(1-2+3-4+\dots+①-1) =$$

$$-d(1-2+3-4+\dots+①-1) = -d\sum_{i=1}^{①-1}(-1)^{i+1}i.$$

The sum of the alternating series in brackets can be easily evaluated. We may for instance obtain it from the corresponding, complete alternating series:

$$1-2+3-4+\dots+(①-1)-①,$$

whose sum is the difference between the sum of the odd numbers in $\mathbb{N}$ and the sum of the even numbers in $\mathbb{N}$. Associating the $①/2$ consecutive pairs in the last summation, we can reduce it to a sum of $①/2$ terms, each equal to -1. The su of this last series is $-①/2$ and the sum of the series we are interested in is, as a consequence, $-①/2+① = ①/2$. It now follows that:

$$-d\sum_{i=1}^{①-1}(-1)^{i+1}i = -d\frac{①}{2}.$$

It may be observed that, for any integer value of d, $-d\frac{①}{2}$ is not expressible in base ten. This highlights yet another computatinal limitation of standard arithmetical methods, which treat the alternating series $a-a+a-\dots$ as having indeterminate value[1]. From our standpoint, the issue of indeterminateness does not arise, as soon as we can provide the length of the series

[1] In more sophisticated contexts, the notion of summation is redefined in order to allow the specification of a numerical value for the sum of the series. An instance of this approach will be discussed at the end of this chapter.

we are interested in, which is never possible in base ten.

Exercise 26.

a) Find the sum of the alternating series associated with $\mathsf{A}'_{1,1}$ making use of Exercise 22, parts (e) and (f).

b) Find the sum of the alternating series associated with $\mathsf{A}'_{2,1}$ e $\mathsf{A}'_{1,3}$.

c) The sum of the complete sequences $a_1,\ldots,a_{①}$ and $b_1,\ldots,b_{①}$ is the sequence $a_1+b_1,\ldots,a_{①}+b_{①}$. Moreover, if A is the complete sequence $a_1,\ldots,a_{①}$, then $-\mathsf{A}$ is the complete sequence $-a_1,\ldots,-a_{①}$.
   - Describe the complete sequences $\mathsf{A}_{1,2}+\mathsf{A}'_{2,3}$, $\mathsf{A}_{2,2}+\mathsf{A}'_{1,3}$ e $\mathsf{A}'_{1,2}-\mathsf{A}_{3,1}$.
   - Find the sum of the series associated with each of the above sequences.
   - Find the sum of the series associated with the sequence $-(\mathsf{A}_{1,1}-\mathsf{A}'_{2,2}+\mathsf{A}'_{2,3})$.

d) Find the sum of the alternating series of even numbers in $\mathbb{N}$.

e) Find the sum of the alternating series of odd numbers in $\mathbb{N}$.

3.2.2 Series with infinitesimals

So far we have considered series with terms in $\mathscr{Z}$. We now lift the restriction. A case of special interest occurs when all the terms of a series are infinitesimals in $\mathscr{Q}$. For example, setting $a = 1/①$, the complete sequence of coefficient a determines the series:

$$\underbrace{\frac{1}{①}+\ldots+\frac{1}{①}}_{①\ \textit{volte}} = 1,$$

whose sum is a finite number.

Exercise 27.

a) Determine the length of a constant sequence of coefficient $1/①^2$ associated with a series whose sum is smaller than $1/n$, for each n expressible in base ten.

b) Determine the length of a constant sequence of coefficient $1/①$ associated with a series whose sum is greater than ①.

c) Describe a constant sequence of infinitesimal coefficient associated with a series whose sum is an infinitesimal.

d) Determine the length of an extension of the constant sequence from (c) such that the sum of its associated series is greater than ①.

e) Describe a complete sequence devoid of infinitely large terms and with an associated a series whose sum is $(① + 1)/2$.

f) Describe a complete sequence devoid of infinitely large terms and with an associated a series whose sum is $①/4$.

Section 3.2.1 places us in a position to insert infinitesimals into series that we already know how to sum. Consider for example the arithmetical progression $\mathsf{A}_{a,d}$. The sum of the corresponding series can be expressed in general terms without invoking the restriction that a, d should be in $\mathscr{Z}$. They may well both be in $\mathscr{Q}$. Thus, there is no problem finding the sum of the series associated with, say, $\mathsf{A}_{1,2/①}$. It is:

$$\sum_{i=0}^{①-1} 1 + \frac{2i}{①} = ① + (① - 1) = 2① - 1.$$

If $a = 1/①$ and $d = 2/①$, the sum of the series associated with $\mathsf{A}_{a,d} =$

$\mathsf{A}_{\frac{1}{\text{①}},\frac{2}{\text{①}}}$ is, as may be easily verified, 1.

Exercise 28.

a) Find positive values of a,d such that the sum of the series associated with $\mathsf{A}_{a,d}$ is smaller than $1/\text{①}^3$.

b) Compute the sum of the series associated with the extension of $\mathsf{A}_{\frac{1}{\text{①}},\frac{1}{\text{①}2}}$ that has length ①^3.

c) Compute the sum of the series associated with $\mathsf{A}'_{1,\frac{2}{\text{①}}}$.

d) Compute the sum of the series associated with $\mathsf{A}'_{\text{①},\frac{2}{\text{①}}}$.

e) Compute the sum of the series associated with $\mathsf{A}'_{\text{①},\frac{1}{\text{①}4}}$.

3.2.3 Geometric series

Given $a > 0$, the complete sequence:

$$a, a^2, a^3, \ldots, a^{\text{①}-1}, a^{\text{①}}$$

whose terms are the first ① positive powers of a is a *complete geometric progression of ratio a*, since the ration of two consecutive terms (the later over the earlier) is constant and equal to a. The associated series is the **complete geometric series**:

$$\sum_{i=1}^{\text{①}} a^i = a^1 + a^2 + \ldots, +a^{\text{①}}.$$

Assuming $a \neq 1$ and letting S be the sum of the last series, it is not difficult to see that $aS = S - a + a^{\text{①}+1}$. There follows that:

$$S = \frac{a^{\text{①}+1} - a}{a-1} = \frac{a}{a-1}(a^{\text{①}} - 1).$$

We apply this general result to the special case $a = \text{①}$. Then:

$$\sum_{i=1}^{\text{①}} \text{①}^i = \frac{\text{①}}{\text{①}-1}(\text{①}^{\text{①}} - 1).$$

If, on the other hand, $a = 1/\text{①}$, we obtain:

$$\sum_{i=1}^{\text{①}} \left(\frac{1}{\text{①}}\right)^i = \frac{1}{1-\text{①}}\left(\frac{1}{\text{①}^{\text{①}}} - 1\right),$$

which is the product of two negative factors and must be, as a consequence, positive, as expected from a sum of positive terms.

Exercise 29.

a) Given a complete geometrical progression of ratio a, let S the sum of the corresponding geometric series. Compute S/a and determine the value of S in terms of a.

b) Given he geometric progression $a_i = 1/2^i\,;\ i = 1, 2, \ldots, \text{①}$, determine the sum of the corresponding geometric series.

c) Compute $\sum_{i=1}^{\text{①}} \frac{1}{3^i}$.

d) Compute $\sum_{i=1}^{\text{①}} \frac{1}{5^i}$.

We do not have to restrict ourselves to complete sequences. We are, for instance, in a position to compute:

$$\sum_{i=1}^{\text{①}-3} \frac{1}{4^i}$$

in at least two different ways. We may find the sum of the corresponding, complete series and then subtract the last three terms of this series from the result or, less circuitously, we can use a general formula for the sum of a geometric series with $\text{①} - 3$ terms. If we adopt the first strategy, we first obtain:

$$\sum_{i=1}^{\text{①}} \frac{1}{4^i} = \frac{1}{3}\left(1 - \frac{1}{4^{\text{①}}}\right) = \frac{1}{3}\left(\frac{4^{\text{①}} - 1}{4^{\text{①}}}\right).$$

Next, we subtract the terms $1/4^i$ with $i = \text{①} - 1, \text{①} - 1, \text{①}$ from the last sum. The result is:

$$\sum_{i=1}^{\text{①}-3} \frac{1}{4^i} = \frac{1}{3}\left(\frac{4^{\text{①}} - 1}{4^{\text{①}}}\right) - \left(\frac{1}{4^{\text{①}-2}} + \frac{1}{4^{\text{①}-1}} + \frac{1}{4^{\text{①}}}\right),$$

which, with some simplifications, leads to the final result:

$$\begin{aligned}
\sum_{i=1}^{①-3}\frac{1}{4^i} &= \frac{1}{3}\left(\frac{4^{①}-1}{4^{①}}\right)-\left(\frac{1}{4^{①-2}}+\frac{1}{4^{①-1}}+\frac{1}{4^{①}}\right)\\
&= \frac{1}{4^{①-2}}\left[\frac{4^{①}-1}{48}-\left(1+\frac{1}{4}+\frac{1}{16}\right)\right]\\
&= \frac{1}{4^{①-2}}\left[\frac{4^{①}-1}{48}-\frac{63}{48}\right]\\
&= \frac{1}{4^{①-2}}\left[\frac{4^{①}-64}{48}\right]\\
&= \frac{1}{4^{①-2}}\left[\frac{4^{①-2}-4}{3}\right]\\
&= \frac{1}{4^{①-3}}\left[\frac{4^{①-3}-1}{3}\right]
\end{aligned}$$

On the other hand, setting $a = 1/4$, we can repeat the argument at the beginning of this section on the sum S of the series we are interested in, noting that its length is $① - 3$. We obtain:

$$\sum_{i=1}^{①-3}\frac{1}{4^i} = \frac{\frac{1}{4}}{\frac{1}{4}-1}\left(\frac{1}{4^{①-3}}-1\right) = \frac{1}{3}\left[\frac{4^{①-3}-1}{4^{①-3}}\right] = \frac{1}{4^{①-3}}\left[\frac{4^{①-3}-1}{3}\right].$$

Exercise 30. Compute the sum of each of the following series:

a) $\sum_{i=1}^{①-2}\frac{1}{2^i}$;

b) $\sum_{i=1}^{\frac{①}{2}}\frac{1}{3^i}$;

c) $\sum_{i=1}^{①-4}\frac{1}{5^i}$;

d) $\sum_{i=1}^{①^2}\frac{1}{5^i}$;

e) $\sum_{i=1}^{\frac{①}{2}-1}\frac{1}{3^i}$;

f) $\sum_{i=1}^{2①}\frac{1}{2^i}$;

The reader familiar with the theory of infinite series knows that only the geometric series of ratio a with $-1 < a < 1$ have a sum expressible in a standard numeral system. We have already seen that working in base ①,

$a < 1$ is unnecessary and that, in addition, we may choose for a values between -1 and 1 that are not expressible in base ten. We now show that, even if $a < 0$, provided $a \neq -1$ (for $a = -1$ we obtain a variant of Grandi's series). When a is negative, we in essence confront the problem of summing an alternating geometric series. It is convenient to set $a = -c$, with $0 < c$, in order to rewrite the expression

$$a, a^2, a^3, \ldots, a^{①-1}, a^{①}$$

in the form:

$$-c, c^2, -c^3, \ldots, -c^{①-1}, c^{①}.$$

The terms of even exponent are positive, while those of odd exponent are negative. After associating the terms with an equal sign, we arrive at:

$$(c^2 + c^4 + \ldots + c^{①-2} + c^{①}) - c(c^0 + c^2 + +c^4 + \ldots + c^{①-4} + c^{①-2}) = S - cT,$$

where each set of brackets includes $①/2$ summands. We compute the sum of the even powers of c. Setting:

$$S = \sum_{i=1}^{\frac{①}{2}} c^{2i}$$

we observe that:

$$c^2 S = S - c^2 + c^{①+2} \quad \text{da cui} \quad S = \frac{c^{①+2} - c^2}{c^2 - 1} = c^2 \left(\frac{c^{①} - 1}{c^2 - 1} \right).$$

Since $T = 1 + S - c^{①}$, the sum of the whole series must be:

$$S - cT = S - c(1 + S - c^{①}) = (1 - c)S + c^{①+1} - c = \frac{c^{①+1} - c}{c+1} = \frac{c}{c+1}(c^{①} - 1).$$

As a concrete illustration, let us consider the case $a = -2$ (thus, $c = 2$). We obtain:

$$\sum_{i=1}^{①} (-2)^i = \frac{2}{3}(2^{①} - 1).$$

The numerical result may induce an initial feeling of uncertainty. Since only terms in $\mathscr{Z}$ are being summed, the result must be an integer. The factor $2/3$ seems to be at variance with this expectation, especially because it multiplies an integer. The problem would be solved if we could show that $2^{①} - 1$ is a multiple of 3. We show that, in $\mathscr{N}$, the immediate precedessor of an even power of 2 is a multiple of 3. More precisely, we show that, for each $k \in \mathcal{N}$ there is $m \in \mathcal{N}$ such that $2^{2k} - 1 = 3m$. We note that the result certainly holds when $k = 1$. Next, we suppose that it holds at k and use this hypothesis to show that it must also hold at $k+1$ (this is, in mathematical jargon, a proof by inducion). Let us therefore assume that $2^{2k} - 1 = 3m$ holds. Equivalently, $2^{2k} = 3m+1$ holds. Multiplying both sides of the last equation by $2^2 = 4$, we obtain:

$$2^2 2^{2k} = 12m+4 \quad \text{che equivale a} \quad 2^{2(k+1)} = 12m+4.$$

Clearly $2^{2(k+1)} - 1 = 12m+3$ is a multiple of 3. This completes our argument. A special case of the argument, for $k = ①/2$, enables us to deduce that $2^{①} - 1$ is a multiple of 3.

Exercise 31. Find the sum of each of the following geometric series:

a) $\sum_{i=1}^{①} (-3)^i$;

b) $\sum_{i=1}^{①} \left(-\frac{1}{3}\right)^i$;

c) $\sum_{i=1}^{①^2} (-①)^i$;

d) $\sum_{i=1}^{\frac{①}{3}} \left(-\frac{1}{4}\right)^i$.

3.3 The arithmetic of infinite series

We conclude this chapter with some miscellaneous considerations on infinite series. Our discussion of series has revolved around an examination of the computational advantages delivered by the introduction of the numeral system in base ①. We have seen that, in many circumstances, a lack of numerical discriminability between distinct series or the computational inaccessibility of their sums typical of standard techniques are easily overcome in terms of the arithmetical calculus from Chapter 2. On account of its applied nature, this calculus does not only enable computations with infinitely large or small terms, but also numerical evaluations of size, which prove of decisive help when the size of interest is the length of a series or the number of its terms. In the absence of an applied treatment of size in a calculus of quantities that are not only finite, series are only partially comparable and, in fact, only their finite heads allow an explicit comparisons, while nothing may be said of their tails. Two generic series A, B presented in the standard forms:

$$A = a_1 + a_2 + a_3 + \dots \quad \text{e} \quad B = b_1 + b_2 + b_3, \dots.$$

do not have numerically specifiable lengths, if the specifications must be found in, say, a base ten numeral system. This lack of numerical determinacy gives rise to a number of complications. While the new series $A+B$ has a natural definition as $(a_1+b_1)+(a_2+b_2)+\ldots$, it is not in general guaranteed (absolute convergence is to be invoked to ensure it), that $A+B=B+A$ holds. The product $A \cdot B$, by contrast, allows at least two natural definitions, whose equivalence is not, however, a matter of course, but can be ensured only under special, additional conditions are introduced. Finally, the ratio A/B, when $B \neq 0$, usually eludes examination in standard terms. We are in a position to bypass all of these difficulties at once because we do not encounter cases in which A, B are values that can give rise to indeterminate forms or violate ordinary arithmetical properties. This saves us from having to apply arithmetical operations to series through a reliance on limits and new definitions of special arithmetical operations. It also spares us to have to resort to supplementary conditions to ensure that series can be operated upon arithmetically in the ordinary way. For an illustration of the last remarks, consider the complete series:

$$A=\sum_{i=1}^{①}\left(\frac{5}{①}\right)^{i}=\frac{5}{5-①}\left(\frac{5^{①}-①^{①}}{①^{①}}\right) \quad \text{e} \quad B=\sum_{i=1}^{①}(-5)^{i}=\frac{5}{6}(5^{①}-1).$$

In base ten, the summands of A are not even expressible. Those of B can, but its sum cannot. It is clear that, under these conditions, the very problem of finding the ration of B over A is not statable. While expanding the range of problems we can tackle, we also simplify their numerical treatment. In the present case, we can easily evaluate:

$$\frac{B}{A}=\frac{①^{①-1}①(5-①)(5^{①}-1)}{6(5^{①}-①^{①})}.$$

Besides having to restrict attention to manageable problems relative to a fixed numeral system, the traditional approach to series is affected by the additional trouble that its problems change their features if they are restated in seemingly equivalent ways. A distinctive lack of control over equivalent presentations is immediately evinced by the standard presentation of Grandi's series:

$$1-1+1-1+1-1+\ldots.$$

If consecutive pairs of terms can be associated, it is legitimate to conclude that the sum of the series should be 0. If, however, it is legitimate to rearrange terms, then the original series has the equivalent form:

$$1+1-1+1+1-1+\dots$$

which no longer determine a sum expressible in base ten. This difficulty essentially relies that operations of rearrangement are not the subject of an applied arithmetical calculus, i.e. they are not numerically determinate. Any rearrangement that does not involve only finitely many terms is in standard form only qualitatively described as the indefinite iteration of some operation. It is impossible to keep track of how many times the operation (say, permuting two consecutive terms) is performed and to check that it can be performed as many times as desired. The principle of Euclidean Size which underlies our point of view demands that we introduce sharper specification than the generic reference to an indefinitely repeatable process. If, for instance, we swap consecutive terms and we are working on a complete sequence, we cannot perform ① swaps, since each swap mobilises two terms and there are only $①/2 < ①$ consecutive pairs of them.

The last observation applies to the study of Grandi's series, once it is numerically specified that it is a complete series (this is how we have defined it above). In this case its sum is zero and stays the same value even after rearrangements. For instance, if we swapped consecutive terms in order to produce blocks of three terms, the first two of which are positive, we run out of positive terms after the first $3①/4$ ones. Since these terms can be subdivided into blocks of three, each of which sums up to 1, Grandi's series truncated at $3①/4$ terms has sum $①/4$. Since the remainder of the series contains $①/4$ negative terms, the sum of Grandi's series continues to be zero, as before.

We have, in fact, a general argument to show that the rearrangement of terms in a complete series does not affect its sum (the same observations apply if the length of the series is known but differs from ①). We consider a generic, complete sequence:

$$a_1, a_2, a_3, \dots, a_{①-1}, a_{①},$$

and take the sum of the corresponding series to be S. To rearrange the ① terms of the series is to permute the terms of the associated sequence and sum them the terms of the permuted sequence. We describe the action of a permutation π on the terms a_i of the complete sequence. We first consider a_1: either π moves it to another position along the sequence, or it does not. If not, the permuted sequence begins with a_1 and we move on to examine a_2. We continue examining successive elements until we reach one that is actually transposed. This may never happen, in which case π

is the identical permutation. If, on the other hand, a_1 is moved by π, then suppose that π tells us to move a_1 to the place occupied by a_i. We swap a_i and a_1. The resulting sequence is:

$$a_i, a_2, \ldots, a_{i-1}, a_1, a_{i+1}, \ldots, a_{①-1}, a_{①}.$$

Now we turn to a_i. If π actually swapped a_1 and a_i, we are done with both and we move on to consider a_2. Otherwise, we swap a_i and a_j, where a_j occupies the place to which π moves a_i. Now a_1, a_i are fixed under the action of π and we have to examine a_j. If a_j is where π prescribes that it had to be moved, we are done and move on to the term of the sequence with least index that needs to be considered. Otherwise we swap a_j with the item in the position that a_j needs to occupy according to π. The procedure just described is a sequence of no more than ① switches, al by the end of which π has been applied. The initial series is:

$$a_1 + (a_2 + \ldots + a_{i-1}) + a_i + (a_{i+1} + \ldots + a_{①}) = a_1 + X + a_i + Y,$$

where X, Y are, respectively, the numerical values of the first and second sum within brackets. After the first switch, say between a_1 and a_i, we obtain:

$$a_i + (a_2 + \ldots + a_{i-1}) + a_1 + (a_{i+1} + \ldots + a_{①}) = a_i + X + a_1 + Y.$$

Using the commutativity and associativity of arithmetical addition, we obtain:

$$S = a_1 + X + a_i + Y = X + (a_1 + a_i) + Y = X + (a_i + a_1) + Y = \\ (X + a_i) + a_1 + Y = a_i + X + a_1 + Y.$$

A single switch does not alter S. This numerical value is preserved by the switches ($\leq$ ①) needed to carry out the permutation π. This result is of interest in view of a classical theorem concerning series that was independently proved by Bernhard Riemann (1826–1866) and Ulisse Dini (1845–1918). In light of the theorem, certain special series, known as conditionally convergent series, can, if properly rearranged, sum to any real number (including the 'extended' values $-\infty$ and $+\infty$). The result is not in conflict with our argument, because it is obtained under a different arithmetical régime, where the length of a series cannot be numerically determined and its summands are neither infinitely large nor infinitely small. On account of its critical dependence on fixed numeral resources, the theorem established by Riemann and Dini is not required to hold if alternative numeral resources are in employment. This is what happens in our case. A typical illustration of the result of Riemann and Dini is offered by the alternating harmonic series:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\dots\ .$$

Anatoly Zhigljavsky has shown in [**37**] that, as soon as this series is treated as a complete one, its sum is unaffected by rearrangements. The reader interested in a more formal discussion of the rarrangement of series may, beside [**37**], turn to the axiomatic approach in [**18**] (other noteworthy results concerning series are contained in [**30**]).

We close with a discussion of alternative ways of summing series. In the face of problems posed by series like $1+2+3+\dots$ or Grandi's series, alternative techniques of summation have been introduced, under which unique numerical values can be established. In the face of a computational problem, one may change the technique of computation or the numeral resources. The latter change is somewhat subtler, because it calls for a more radical departure from habit than the introduction of techniques built from known resources does. The former change requires much ingenuity, but does not require forging new resources. Its interest lies for us in its connection with the approach we have been pursuing. We establish a connection through a concrete illustration, namely the problem of finding a numerical value for the infinite series:

$$1+2+3+4+\dots$$

as it was tackled by S̀rinivasa Ramanujan (1887–1920). Calling S the sum of the series, we observe that, since $S=4S-3S$, we may, following Ramanujan, express $-3S$ as the difference $S-4S$ as follows:

$$\begin{array}{rcccccccccccc} S &=& 1 &+& 2 &+& 3 &+& 4 &+& 5 &+& 6 & \dots \\ -4S &=& & - & 4 & & & - & 8 & & & - & 12 & \dots \\ -3S &=& 1 &-& 2 &+& 3 &-& 4 &+& 5 &-& 6 & \dots . \end{array}$$

Ramanujan then relied on the fact that, for $x=1$, the formal power series of $1/(1+x^2)$ (which is $1/4$ when $x=1$) determines the alternating series $-3S$. He could conclude:

$$-3S=\frac{1}{4}\ \text{, which implies}\quad S=-\frac{1}{12}.$$

Rather strikingly, a series of positive terms is assigned a negative sum. The value we found was, by contrast, positive and infinitely large: it was

$①(①+1)/2$. We now show that this value can be recovered following Ramanujan's strategy, as long as we specify the numerical length of the series involved (in this case it becomes unnecessary to turn to a formal power series). Working with the complete series $1+2+\ldots+(①-1)+①$, we note that, in computing $S-4S$ as Ramanujan proposed, we have to shift $①/2$ summands. As inspection shows:

$$\begin{array}{rcccccccccccccc} S & = & 1 & + & 2 & + & 3 & \ldots & + & ① & & & & & \\ -4S & = & & - & 4 & & & \ldots & - & 4\frac{①}{2} & - & 4\left(\frac{①}{2}+1\right) & - & \ldots & - \quad 4① \\ -3S & = & 1 & - & 2 & + & 3 & - & \ldots & -① & - & 4\left(\frac{①}{2}+1\right) & - & \ldots & - \quad 4①. \end{array}$$

Infinitely many terms from the sum $-4S$ are not summed with a positive term from the row above them. For this reason, the proper way to evaluate $-3S$ is to consider the extended series:

$$-3S = (1-2+3-4+\ldots+(①-1)-①)-4\left(\left(\frac{①}{2}+1\right)+\ldots+①\right) = X-4Y$$

We already know that $X=-①/2$ and we can exploit the fact that the terms in Y are the first half of the sequence $\mathsf{A}_{\frac{①}{2},1}$ to compute:

$$Y = \frac{①^2}{4}+\frac{①}{4}\left(\frac{①+2}{2}\right).$$

Finally:

$$-3S = -\frac{①}{2}-①^2-\frac{①(①+2)}{2} = \frac{-①-2①^2-①^2-2①}{2} = -3\frac{①^2+①}{2}.$$

It is easy to see that:

$$S = \frac{①(①+1)}{2}.$$

This treatment of Ramanujan's summation is due to Yaroslav Sergeyev, see **[34]**.

4. How to use the rest of this textbook

We observed in Chapter 1 that the arithmetical calculus developed so far has been used in manifold applications, both for the sake of refining the treatment of existing problems, solving given, open problems and, finally to formulate and tackle new problems. Many open problems referring to infinitely large and small quantities in qualitative terms have an elementary character: they have resisted resolution because the elementary resources that match their character are not those of traditional arithmetic but rather the resources of the applied arithmetical calculus with which we have been working until now. In the absence of an adequate elementary treatment, the problems in question arise as **paradoxes of infinity**. Their resolution consists in the supply of methods suitable to relate them to otherwise unavailable numerical computations. Paradoxes are thus best viewed not as persisting puzzles but as requests of new mathematical methods.

Six paradoxes of infinity, each of which admits a number of variations, are presented in the following chapter. Each is introduced in a self-contained manner and can be studied without referring to the rest of the textbook (this is why the introduction to each paradox consists of the same background material). There is no need to work through each paradox. Individual ones, of special interest, may be selected for individual or group work among students or even for joint work involving both students and teachers in the same learning process.

The exploration of each paradox or family of paradoxes is structured through a progression of exercises intended to promote both problem-solving skills and the less familiar practice of criticising the statement of a problem and reconstructing it in terms amenable to resolution.

Exercises are partitioned into two categories: starred ones call for an application of the arithmetical calculus based on ①; unstarred ones are to be carried out under a restriction of numeral resources to those available in base ten.

5. Worksheet A: Hilbert's hotel

5.1 Counting $\mathbb{N}$

Let us suppose that five objects are given to us and that we wish to count them. As we count them, we assign to them the symbols $1,2,3,4,5$. These numerical symbols are not the only ones at our disposal, but belong to the sequence $1,2,3,4,5,\ldots$, which we can extend as far as we please. We may thus describe a count of five items as follows:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\ 1 & 2 & 3 & 4 & 5. & & & \end{array}$$

A completed count is nothing but the application of an initial segment of a counting system to a collection of items. We extend this idea to infinite collections. In particular, we want to describe a completed count of $\mathbb{N}$, the collection of all items $1,2,3,\ldots$ in such a way that it can be carried out exactly as the count of five objects we just described can. **To this end, we introduce a new numerical notation that enables us to express a completed count of $\mathbb{N}$ as a specific initial segment of a counting system**. The latter completed count ends with ① (*gross-one*), which:

1. follows every number expressed in the ordinary (base ten) notation. Thus ① $> 1,2,3,4,5,6,\ldots$;
2. behaves, from the point of view of arithmetical calculations, exactly like the numbers we are used to dealing with.

A completed count of positive integers looks like this:

$$\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & \dots & ①-2 & ①-1 & ① & ①+1 & ①+2 & \dots \\ 1 & 2 & 3 & 4 & 5 & \dots & ①-2 & ①-1 & ①. & & & \end{array}$$

The new numerical notation we have introduced allows us to identify:

$$1,2,3,\dots,①-1,①$$

as the initial segment of a more extensive system of numerical symbols.

Exercise 32.

a) Making use of (1), verify that $①-1 > 1,2,3,4,\dots$ and that $①-2 > 1,2,3,4,\dots$. Generalise these results.

b) Making use of (2), explain why the following inequalities hold: $①-1 < ①$ e $① < ①+1$.

c) Making use of (2), compute (i) $①-①$ and (ii) $4①-①-3①+3$.

d) Making use of (2), compute $(①+1)^2$.

5.2 Counting subdivisions of $\mathbb{N}$

We can describe $\mathbb{N}$ as the following sequence:

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad \dots$$

If we rewrite the same sequence on two columns, moving from top to bottom and from left to right, we obtain:

$$\begin{array}{cccccc} 1 & 3 & 5 & 7 & 9 & \dots \\ 2 & 4 & 6 & 8 & 10 & \dots \end{array}$$

We have split $\mathbb{N}$ into two equal parts, the odd and the even numbers. We obtain a finer subdivision, into three equal parts, by rewriting the initial sequence on three columns, moving again from top to bottom and from left to right:

$$\begin{array}{cccccc} 1 & 4 & 7 & 10 & 13 & \dots \\ 2 & 5 & 8 & 11 & 14 & \dots \\ 3 & 6 & 9 & 12 & 15 & \dots \end{array}$$

Since there are ① numbers in $\mathbb{N}$, the subdivision into two equal parts splits it into $①/2$ even numbers and $①/2$ odd numbers. The subdivision into

three equal parts splits it into three sequences, each of which contains ①/3 numbers.

Exercise 33.

a) Using ①, we can describe ℕ thus:

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \dots \quad ① - 2 \quad ① - 1 \quad ①.$$

Since ①/2 + ①/2 = ①, ① is even. By (2), ① − 1 is odd and ① − 2 is even. Thus, the sequence of even numbers in ℕ is:

$$2 \quad 4 \quad 6 \quad 8 \quad 10 \quad \dots \quad ① - 4 \quad ① - 2 \quad ①,$$

which cannot be as long as the full sequence ℕ, but is only half as long. Compare the sequence of the even numbers in ℕ with a completed count of ℕ.

b) Compare the sequence of the odd numbers in ℕ with a completed count of ℕ. Compare the sequence of the multiples of 3 in ℕ with a completed count of ℕ.

c) Using ①, describe the subdivision of ℕ into four equal parts. How many numbers are there in each part of this subdivision?

5.3 Hilbert's hotel with one new guest

Note: *some of the following exercises are starred. These exercises are to be solved by making use of ①. The other, unstarred exercises are to be solved without making use of ①.*

Hilbert's hotel has infinitely many rooms. More precisely, these rooms can be counted by a completed count of ℕ. Emmy would like to book a room, but she is told that the hotel is currently full. After giving the problem a thought, Emmy proposes a way of freeing up a room, without having any of the current guest leave the hotel. She points out that if the guest in room 1 moves to room 2, the guest in room 2 to room 3, and so on, then she can stay in room 1, whilst every other guest will have a new room.

Exercise 34.

a) If Emmy is right, the same number of single guests can equally well take all rooms in Hilbert's hotel or take all but one rooms without sharing. By the same clue, the very same number of guests can take all but two rooms without sharing. Generalise these remarks and discuss their correctness.

b) *Is it correct to speak of the *number* of guests? Can this number be represented? Under what conditions?

c) Emmy's rule can be described a correspondence between old and new room numbers, as follows:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \dots \\ 2 & 3 & 4 & 5 & 6 & \dots \end{array}$$

in each column, the number at the top designates the old room of a specific guest and the number at the bottom designates the new room the guest is to occupy. The assignment looks right: why? Does it make sense to say that this assignment will, sooner or later, be completed (i.e. each new room will be assigned to a specific guest)?

d) *Make use of ① to evaluate Emmy's proposal. In particular, specify the number of guests before Emmy's arrival and the number of (prospective or actual) guests after her arrival.

5.4 Hilbert's hotel with infinitely many new guests

ilbert's hotel has infinitely many rooms. More precisely, these rooms can be counted by a completed count of $\mathbb{N}$. Emmy has so many friends that they can only be counted by a completed count of $\mathbb{N}$. She would like to stay at Hilbert's hotel with them, but the hotel is currently full. After thinking about it for a bit, Emmy proposes a way of finding rooms for herself and all of her friends, without dislodging any of the current guests. First, the guest in room 1 should move to room 2, the guest in room 2 should move to room 4, the guest in room 3 should move to room 6, and so on. Emmy can stay in room 1 and her infinitely many friends can take rooms $3, 5, 7, \ldots$.

Exercise 35.

a) Felix has as many friends as Emmy does and he arrives at Hilbert's hotel exactly when Emmy does. What might Emmy propose to Felix in order to ensure that they and their friends should find suitable accommodation at Hilbert's hotel without dislodging any of the current guests (the hotel is, of course, currently full)?

b) *How many rooms are needed in order to accommodate Emmy and her friends? How many rooms does Emmy's plan free up? Find the number of guests lacking a room after Emmy's plan is put into practice.

c) *Repeat the last exercise with respect to Emmy's plan as you described it in (a).

6. Worksheet B: Thomson's lamp

6.1 Counting $\mathbb{N}$

Let us suppose that five objects are given to us and that we wish to count them. As we count them, we assign to them the symbols $1,2,3,4,5$. These numerical symbols are not the only ones at our disposal, but belong to the sequence $1,2,3,4,5,\ldots$, which we can extend as far as we please. We may thus describe a count of five items as follows:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\ 1 & 2 & 3 & 4 & 5. & & & \end{array}$$

A completed count is nothing but the application of an initial segment of a counting system to a collection of items. We extend this idea to infinite collections. In particular, we want to describe a completed count of $\mathbb{N}$, the collection of all items $1,2,3,\ldots$ in such a way that it can be carried out exactly as the count of five objects we just described can. **To this end, we introduce a new numerical notation that enables us to express a completed count of $\mathbb{N}$ as a specific initial segment of a counting system**. The latter completed count ends with ① (*gross-one*), which:

1. follows every number expressed in the ordinary (base ten) notation. Thus $① > 1,2,3,4,5,6,\ldots$;
2. behaves, from the point of view of arithmetical calculations, exactly like the numbers we are used to dealing with.

A completed count of positive integers looks like this:

$$\begin{array}{ccccccccccccc} 1 & 2 & 3 & 4 & 5 & \dots & ①-2 & ①-1 & ① & ①+1 & ①+2 & \dots \\ 1 & 2 & 3 & 4 & 5 & \dots & ①-2 & ①-1 & ①. & & & \end{array}$$

The new numerical notation we have introduced allows us to identify:

$$1,2,3,\dots,①-1,①$$

as the initial segment of a more extensive system of numerical symbols.

Exercise 36.

a) Making use of (1), verify that $①-1 > 1,2,3,4,\dots$ and that $①-2 > 1,2,3,4,\dots$. Generalise these results.

b) Making use of (2), explain why the following inequalities hold: $①-1 < ①$ e $① < ①+1$.

c) Making use of (2), compute (i) $①-①$ and (ii) $4①-①-3①+3$.

d) Making use of (2), compute $(①+1)^2$.

Exercise 37.

a) Property (2) also applies to arithmetic with negative numbers and fractions. For instance:

$$\frac{①}{4}-\frac{①}{3}=-\frac{①}{12};\ \left(\frac{1}{2}\right)^{①}=\frac{1}{2^{①}}.$$

Compute:

$$\text{(i) } \frac{3①}{4}-\frac{①}{2};\ \text{(ii) } \frac{1}{2^{①}}+\frac{1}{2^{①}}$$

6.2 Summing powers of $1/2$

Consider the sum of the first four positive powers of $\frac{1}{2}$:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}.$$

One way to compute it is to set its value equal to x and verify that:

$$2x = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} = 1+x-\frac{1}{16}.$$

The sum we are looking for is $x = 1 - \frac{1}{16} = 1 - \frac{1}{2^4}$.

Exercise 38.

a) Using the argument just provided, find the sum of the first n positive powers of $1/2$.

b) Compute:

$$\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{①}}.$$

c) Compute:

$$\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{①-1}}.$$

6.3 Even and odd numbers

We can describe $\mathbb{N}$ as the following sequence:

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad \ldots$$

If we rewrite the same sequence on two columns, moving from top to bottom and from left to right, we obtain:

$$\begin{array}{cccccc} 1 & 3 & 5 & 7 & 9 & \ldots \\ 2 & 4 & 6 & 8 & 10 & \ldots \end{array}$$

We have split $\mathbb{N}$ into two equal parts, the odd and the even numbers. Since there are ① numbers in $\mathbb{N}$, the subdivision into two equal parts splits it into $①/2$ even numbers and $①/2$ odd numbers. We have:

$$① = \frac{①}{2} + \frac{①}{2},$$

where $①/2$ is a whole number in a completed count of $\mathbb{N}$. It is possible to verify that ①$/2 > 1, 2, 3, 4, \ldots$. The last equality implies ①$/2 <$ ①.

Exercise 39.

a) Using ①, we can describe $\mathbb{N}$ by the completed count:

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \ldots \quad ①-2 \quad ①-1 \quad ①.$$

Since ①$/2+$①$/2 = 1$, ① is even. By (2), ①-1 must be odd and ①-2 even. Thus, the sequence of even numbers in $\mathbb{N}$ is:

$$2 \quad 4 \quad 6 \quad 8 \quad 10 \quad \ldots \quad ①-4 \quad ①-2 \quad ①,$$

but it cannot be as long as $\mathbb{N}$ because it contains half the numbers in it. Compare the sequence of the even numbers in $\mathbb{N}$ with a completed count of $\mathbb{N}$.

b) Compare the sequence of the odd numbers in $\mathbb{N}$ with a completed count of $\mathbb{N}$.

6.4 Thomson's lamp

Note: *some of the following exercises are starred. These exercises are to be solved by making use of ①. The other, unstarred exercises are to be solved without making use of ①.*

A lamp[1], at first off, is switched on after $1/2$ minutes. It is switched off after $1/4$ more minutes. After another $1/8$ minutes, it it switched on again. The lamp is switched on and off as many times as there are items in a

[1] The problem discussed here was originally stated in [**35**]

completed count of $\mathbb{N}$.

Exercise 40.

a) How long does it take to carry switch the lamp on and off 1000 times?

b) How long does it take to carry switch the lamp on and off 1000000 times?

c) Show that, given a specific number, it takes less than a minute to switch the lamp on an off that exact number of times.

Exercise 41. Evaluate the assumptions on which the following argument is based:

> If the lamp is switched on before one minute has passed, it will also be switched off before one minute has passed. Analogously, if the lamp is off before one minute has passed, it will be switched on before one minute has passed. We are forced to conclude that the state (i.e. on or off) of the lamp after one minute is completely indeterminate.

Exercise 42.

a) *How many times is Thomson's lamp switched on and off?

b) *How long does it take to finish switching it on and off? How long does it take to complete half of this process?

c) *Determine the state of the lamp (on or off) after the last but one switch is performed.

d) *Let us extend the number of operations and require that Thomson's lamp be switched on an off after ① + 100 operations? If it is initially off, what is its state after all operations are performed?

e) *What happens to a lamp initially off after 3① + 1 operations?

f) *Is it possible to modify the sequence of switches in such a way that they are completed in exactly one minute?

7. Worksheet C: Arsenjevic's cube

7.1 Counting $\mathbb{N}$

Let us suppose that five objects are given to us and that we wish to count them. As we count them, we assign to them the symbols $1,2,3,4,5$. These numerical symbols are not the only ones at our disposal, but belong to the sequence $1,2,3,4,5,\ldots$, which we can extend as far as we please. We may thus describe a count of five items as follows:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\ 1 & 2 & 3 & 4 & 5. & & & \end{array}$$

A completed count is nothing but the application of an initial segment of a counting system to a collection of items. We extend this idea to infinite collections. In particular, we want to describe a completed count of $\mathbb{N}$, the collection of all items $1,2,3,\ldots$ in such a way that it can be carried out exactly as the count of five objects we just described can. **To this end, we introduce a new numerical notation that enables us to express a completed count of $\mathbb{N}$ as a specific initial segment of a counting system**. The latter completed count ends with ① (*gross-one*), which:

1. follows every number expressed in the ordinary (base ten) notation. Thus $① > 1,2,3,4,5,6,\ldots$;
2. behaves, from the point of view of arithmetical calculations, exactly like the numbers we are used to dealing with.

A completed count of positive integers looks like this:

$$\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & \ldots & ①-2 & ①-1 & ① & ①+1 & ①+2 & \ldots \\ 1 & 2 & 3 & 4 & 5 & \ldots & ①-2 & ①-1 & ①. & & & \end{array}$$

The new numerical notation we have introduced allows us to identify:

$$1,2,3,\ldots,①-1,①$$

as the initial segment of a more extensive system of numerical symbols.

Exercise 43.

a) Making use of (1), verify that $①-1 > 1,2,3,4,\ldots$ and that $①-2 > 1,2,3,4,\ldots$. Generalise these results.

b) Making use of (2), explain why the following inequalities hold: $①-1 < ①$ e $① < ①+1$.

c) Making use of (2), compute (i) $①-①$ and (ii) $4①-①-3①+3$.

d) Making use of (2), compute $(①+1)^2$.

Exercise 44.

a) Property (2) also applies to arithmetic with negative numbers and fractions. For instance:

$$\frac{①}{4}-\frac{①}{3}=-\frac{①}{12}; \left(\frac{1}{2}\right)^{①}=\frac{1}{2^{①}}.$$

Compute:

$$\text{(i) } \frac{3①}{4}-\frac{①}{2}; \text{ (ii) } \frac{1}{2^{①}}+\frac{1}{2^{①}}$$

7.2 Summing powers of 1/2

Consider the sum of the first four positive powers of $\frac{1}{2}$:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}.$$

One way to compute it is to set its value equal to x and verify that:

$$2x = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} = 1+x-\frac{1}{16}.$$

The sum we are looking for is $x = 1 - \frac{1}{16} = 1 - \frac{1}{2^4}$.

Exercise 45.

a) Using the argument just provided, find the sum of the first n positive powers of $1/2$.

b) Compute:

$$\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{①}}.$$

c) Compute:

$$\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{①-1}}.$$

7.3 Counting subdivisions of ℕ

We can describe ℕ as the following sequence:

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad \ldots$$

If we rewrite the same sequence on two columns, moving from top to bottom and from left to right, we obtain:

$$\begin{array}{cccccc} 1 & 3 & 5 & 7 & 9 & \ldots \\ 2 & 4 & 6 & 8 & 10 & \ldots \end{array}$$

We have split ℕ into two equal parts, the odd and the even numbers. We obtain a finer subdivision, into three equal parts, by rewriting the initial sequence on three columns, moving again from top to bottom and from left to right:

$$\begin{array}{cccccc} 1 & 4 & 7 & 10 & 13 & \ldots \\ 2 & 5 & 8 & 11 & 14 & \ldots \\ 3 & 6 & 9 & 12 & 15 & \ldots \end{array}$$

Since there are ① numbers in ℕ, the subdivision into two equal parts splits it into ①$/2$ even numbers and ①$/2$ odd numbers. The subdivision into

three equal parts splits it into three sequences, each of which contains ①/3 numbers.

Exercise 46.

a) Using ①, we can describe ℕ thus:

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \dots \quad ①-2 \quad ①-1 \quad ①.$$

Since ①/2 + ①/2 = ①, ① is even. By (2), ① − 1 is odd and ① − 2 is even. Thus, the sequence of even numbers in ℕ is:

$$2 \quad 4 \quad 6 \quad 8 \quad 10 \quad \dots \quad ①-4 \quad ①-2 \quad ①,$$

which cannot be as long as the full sequence ℕ, but is only half as long. Compare the sequence of the even numbers in ℕ with a completed count of ℕ.

b) Compare the sequence of the odd numbers in ℕ with a completed count of ℕ. Compare the sequence of the multiples of 3 in ℕ with a completed count of ℕ.

c) Using ①, describe the subdivision of ℕ into four equal parts. How many numbers are there in each part of this subdivision?

7.4 Arsenjevic's cube

Note: *some of the following exercises are starred. These exercises are to be solved by making use of ①. The other, unstarred exercises are to be solved without making use of ①.*

A bricklayer[1] piles up green and red slabs alternately. The bottom slab of the pile is red, it is 1/2 metres thick and it is laid 1/2 minutes after a fixed, initial instant. The second slab is put on top of the first after 1/4 minutes: it is green and 1/4 metres thick. The third slab, which is red and 1/8 metres thick, is put on top of the second after 1/8 minutes. The collection of green and red slabs available to the bricklayer can be counted

[1] The problem discussed in this section was originally stated in [**3**].

by a completed count of $\mathbb{N}$.

Exercise 47.

a) How long does it take to lay the first one thousand slabs on top of one another?

b) *How long does it take to lay the first half of the total number of slabs on top of one another?

c) *When half the work has been carried out, what colour is the top of the pile of slabs?

d) *When one third of the work has been carried out, how tall is the pile of slabs that has been built?

e) *When the work is fully carried out, what colour is the top of the pile of slabs? Is the whole pile a cube of side one metre?

f) Determine the assumptions supporting this argument:

> If a green slab is laid before one minute has passed, it will be covered by a red one before one minute has passed. This remains true if the roles of green and red slabs is interchanged. Therefore, after one minute, the top of the pile of slabs is neither red nor green.

Exercise 48.

a) *What colour is the top of the pile after $\frac{①}{3} - 1$ slabs have been put on top of one another?

b) *What colour is the top of the pile after $\frac{①}{6} + \frac{①}{2}$ slabs have been put on top of one another? How long did it take to pile them up?

We know look at two bricklayers, each handling a separate supply of identical slabs counted by a completed count of $\mathbb{N}$. One of them operates in the exact manner described above, whilst the other works at a slower pace and lays the first slab when the first bricklayer has laid the second and keeps lagging behind by one operation relative to his colleague. The slower bricklayer also stops working when the faster one does. .

Exercise 49. Evaluate the following argument, identifying the assumptions supporting it and any possible gaps in it:

> Even if one bricklayer is slower than the other, they are both done laying all slabs at their disposal after one minute. To illustrate why this must be the case, consider the second slab. The first bricklayer lays it after $1/2+1/4$ minutes, i.e. $1/4$ minutes before one minute has passed. The second bricklayer is able to lay his second slab after $1/2+1/4+1/8=7/8$ minutes, just before one minute elapses. As long as one operation is made by the first bricklayer before one minute has passed, the second bricklayer is able to perform the corresponding operation before one minute elapses. Since, however, each slab laid by the first bricklayer is laid before one minute has passed, the second bricklayer can lay the corresponding slab in time. This means that, after one minute, both bricklayers are finished with all their slabs.

Exercise 50.

a) *In the scenario just described, determine the colour of the slabs on top of each bricklayer's pile.

b) *Suppose that the slower bricklayer lays only one slab when the faster one hais laid two. How to the respective piles of slabs differ after one minute?

We now deal with three bricklayers dealing respectively with ①, ①$/2$ and ①$/3$ slabs (whose colour and thickness vary as in the first scenario we have described). The bricklayer with fewer slabs lays the first after $1/2$ minues from a fixed, initial instant, the second after a further interval of $1/4$ minutes, and so on. The bricklayer with ①$/2$ slabs follows the pace of the first one when handling the first ①$/6$ slabs, but then continues at a slower pace until he disposes of all slabs. He is, in the latter part of the piling process, twice as slow as the first bricklayer. Finally, the bricklayer with ① slabs is, from beginning to end four times slower than the second bricklayer when at the slower pace.

Exercise 51.

a) *How long does it take the fastest bricklayer to finish the job? How long does it take the slower to finish?

b) Describe the pile built by each bricklayer after one minute.

c) *Describe the pile built by each bricklayer after one minute.

d) *The fastest bricklayer replaces the slowest after the latter completed half of his job. How long does it take to finish the job of the slowest bricklayer in this case?

8. Worksheet D: Ross's paradox

8.1 Counting $\mathbb{N}$

Let us suppose that five objects are given to us and that we wish to count them. As we count them, we assign to them the symbols $1,2,3,4,5$. These numerical symbols are not the only ones at our disposal, but belong to the sequence $1,2,3,4,5,\ldots$, which we can extend as far as we please. We may thus describe a count of five items as follows:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\ 1 & 2 & 3 & 4 & 5. & & & \end{array}$$

A completed count is nothing but the application of an initial segment of a counting system to a collection of items. We extend this idea to infinite collections. In particular, we want to describe a completed count of $\mathbb{N}$, the collection of all items $1,2,3,\ldots$ in such a way that it can be carried out exactly as the count of five objects we just described can. **To this end, we introduce a new numerical notation that enables us to express a completed count of $\mathbb{N}$ as a specific initial segment of a counting system**. The latter completed count ends with ① (*gross-one*), which:

1. follows every number expressed in the ordinary (base ten) notation. Thus ① $> 1,2,3,4,5,6,\ldots$;
2. behaves, from the point of view of arithmetical calculations, exactly like the numbers we are used to dealing with.

A completed count of positive integers looks like this:

$$\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & \dots & ①-2 & ①-1 & ① & ①+1 & ①+2 & \dots \\ 1 & 2 & 3 & 4 & 5 & \dots & ①-2 & ①-1 & ①. & & & \end{array}$$

The new numerical notation we have introduced allows us to identify:

$$1,2,3,\dots,①-1,①$$

as the initial segment of a more extensive system of numerical symbols.

Exercise 52.

a) Making use of (1), verify that $①-1 > 1,2,3,4,\dots$ and that $①-2 > 1,2,3,4,\dots$. Generalise these results.

b) Making use of (2), explain why the following inequalities hold: $①-1 < ①$ e $① < ①+1$.

c) Making use of (2), compute (i) $①-①$ and (ii) $4①-①-3①+3$.

d) Making use of (2), compute $(①+1)^2$.

Exercise 53.

a) Property (2) also applies to arithmetic with negative numbers and fractions. For instance:

$$\frac{①}{4}-\frac{①}{3}=-\frac{①}{12};\ 3\frac{①}{10}-①=-\frac{7①}{10}.$$

Compute:

$$\text{(i) } \frac{3①}{4}-\frac{①}{2};\ \text{(ii) } \frac{①}{10}(2①+1)-\frac{①}{5}.$$

8.2 Counting subdivisions of $\mathbb{N}$

We can describe $\mathbb{N}$ as the following sequence:

$$1\quad 2\quad 3\quad 4\quad 5\quad 6\quad 7\quad 8\quad 9\quad \dots$$

If we rewrite the same sequence on two columns, moving from top to bottom and from left to right, we obtain:

$$\begin{array}{cccccc} 1 & 3 & 5 & 7 & 9 & \dots \\ 2 & 4 & 6 & 8 & 10 & \dots \end{array}$$

We have split $\mathbb{N}$ into two equal parts, the odd and the even numbers. We obtain a finer subdivision, into three equal parts, by rewriting the initial sequence on three columns, moving again from top to bottom and from left to right:

$$\begin{array}{cccccc} 1 & 4 & 7 & 10 & 13 & \dots \\ 2 & 5 & 8 & 11 & 14 & \dots \\ 3 & 6 & 9 & 12 & 15 & \dots \end{array}$$

Since there are ① numbers in ℕ, the subdivision into two equal parts splits it into ①/2 even numbers and ①/2 odd numbers. The subdivision into three equal parts splits it into three sequences, each of which contains ①/3 numbers.

Exercise 54.

a) Using ①, we can describe ℕ thus:

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \dots \quad ①-2 \quad ①-1 \quad ①.$$

Since $①/2 + ①/2 = ①$, ① is even. By (2), $① - 1$ is odd and $① - 2$ is even. Thus, the sequence of even numbers in ℕ is:

$$2 \quad 4 \quad 6 \quad 8 \quad 10 \quad \dots \quad ①-4 \quad ①-2 \quad ①,$$

which cannot be as long as the full sequence ℕ, but is only half as long. Compare the sequence of the even numbers in ℕ with a completed count of ℕ.

b) Compare the sequence of the odd numbers in ℕ with a completed count of ℕ. Compare the sequence of the multiples of 3 in ℕ with a completed count of ℕ.

c) Using ①, describe the subdivision of ℕ into four equal parts. How many numbers are there in each part of this subdivision?

Exercise 55.

a) Compare a completed count of ℕ with the tenth part of ℕ containing 2.

b) Compare a completed count of ℕ with the tenth part of ℕ containing 10.

8.3 Ross's paradox

Note: *some of the following exercises are starred. These exercises are to be solved by making use of ①. The other, unstarred exercises are to be solved without making use of ①.*

Felix[1] owns as many ping pong balls as can be counted by a completed count of $\mathbb{N}$. His ping pong balls are marked by the numerical labels $1, 2, 3, \ldots$. Felix takes the balls from number 1 to number 10 out of the box and returns the ball number 1. Next, he takes the balls from number 11 to 20 out of the box and returns ball number 2. He continues taking out the balls from number 21 to number 30 and returning 3. He goes on to consider all labelled ping pong balls. When he later meets his friends Hermann and Emmy, Felix asks them if they can figure out how many ping pong balls were left out of the box after he finished moving them in and out of it.

Exercise 56.

a) Describe the fourth and fifth stage of Felix's procedure. Describe the hundredth stage. Describe a generic stage.

b) Emmy observes that, at each stage, nine balls are taken out of the box. In the end, infinitely many balls are out of the box. Is she right? Justify your answer.

c) Hermann observes that, in each stage one ball is returned into the box. Since there are infinitely many stages and ping pong balls are returned to the box systematically (first ball number one 1, then ball number 2, then ball number 3 and so on), every ping pong ball is in the end returned. None are left out of the box. Is Hermann right? Justify your answer.

Paul, another friend of Felix, offers a third solution to his riddle. Paul considers the first ten stages of the procedure carried out by Felix but imagines that Felix was dealing with three boxes A, B and C. Box A is iniitially full, while B, C are empty. Paul thinks that Felix's procedure must be equivalent to one involving three boxes, in which ten balls at a time are taken out of A and nine of them go into B while the remaining ball goes into C. After this is done ten times, there are 90 balls in B and 10 in C. After it is done one hundred tomes, the process ends with 900 balls in B and 100 in C. If A is,

[1] This problem was originally stated by Littlewood in [**14**] and was later discussed by Ross in [**19**].

in the end, emptied, the number of balls in B is nine times the number of balls in C. It suffices to transfer the contents of C into A to transform the procedure with three boxes into the procedure carried out by Felix. This, Paul concludes, means that both B and A end up containing infinitely many balls, i.e. that there are infinitely many balls outside A and infinitely many balls inside it, in a fixed ratio.

Exercise 57. Discuss Paul's argument and compare it with Emmy's and Hermann's proposals. Which one is the most convincing?

Let us now tackle Felix's problem by numerical means.

Exercise 58.

a) *How many stages does Felix have to go through in order to complete his procedure? Why must they be fewer than ① stages? Why must they be fewer than ①/2?

b) *Which ping pong balls are handled in the penultimate stage? Which ones are handled in the final stage? (identify them by their numerical labels)

c) *How many ping pong balls remain inside the box once Felix is finished? How many, as a result, have been taken out?

Because his friends have eventually found the correct answer to his question, Felix devises a new one for them. He reveals them that he owns not just one, but two identical, infinite collections of labelled ping pong balls. He carried out the procedure his friends have been able to describe correctly on one of these collections, but he did something else with the other collection. In the first instance, he extracted the balls from number 1 to number 9 from the second box, took a marker and wrote a zero next to 1 on the ball number 1, thus turning its label into 10. Then Felix took out the balls from the number 11 to the number 19 and, using a marker relabelled the number 2 as 20. Subsequently, he took out of his box the balls from the number 21 to the number 29 and relabelled the number 3 as 30. He kept going like this until there were ping pong balls available to consider. His question for Emmy, Hermann and Paul is: how many ping pong balls were in the end out of the box, how many were left inside it and which numerical labels marked the ping pong balls outside the box?

Exercise 59.

a) *Felix could only go through ①/9 stages. How many ping pong balls are relabelled in the process? How many keep their original label?

b) *How many ping pong balls never leave the box?

c) *How does Felix's second procedure differ from his first procedure?

The reader familiar with Worksheet B may tackle the following questions as well:

Exercise 60.

a) *How can the stages Felix goes through in each procedure be scheduled so that they are completed within one minute? Compute the exact time it takes to complete them.

b) *How can the stages Felix goes through in each procedure be scheduled so that they are completed in exactly one minute?

9. Worksheet E: infinite decisions

9.1 Counting $\mathbb{N}$

Let us suppose that five objects are given to us and that we wish to count them. As we count them, we assign to them the symbols $1,2,3,4,5$. These numerical symbols are not the only ones at our disposal, but belong to the sequence $1,2,3,4,5,\ldots$, which we can extend as far as we please. We may thus describe a count of five items as follows:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\ 1 & 2 & 3 & 4 & 5. & & & \end{array}$$

A completed count is nothing but the application of an initial segment of a counting system to a collection of items. We extend this idea to infinite collections. In particular, we want to describe a completed count of $\mathbb{N}$, the collection of all items $1,2,3,\ldots$ in such a way that it can be carried out exactly as the count of five objects we just described can. **To this end, we introduce a new numerical notation that enables us to express a completed count of $\mathbb{N}$ as a specific initial segment of a counting system**. The latter completed count ends with ① (*gross-one*), which:

1. follows every number expressed in the ordinary (base ten) notation. Thus ① $> 1,2,3,4,5,6,\ldots$;
2. behaves, from the point of view of arithmetical calculations, exactly like the numbers we are used to dealing with.

A completed count of positive integers looks like this:

$$\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & \dots & ① - 2 & ① - 1 & ① & ① + 1 & ① + 2 & \dots \\ 1 & 2 & 3 & 4 & 5 & \dots & ① - 2 & ① - 1 & ①. & & & \end{array}$$

The new numerical notation we have introduced allows us to identify:

$$1, 2, 3, \dots, ① - 1, ①$$

as the initial segment of a more extensive system of numerical symbols.

Exercise 61.

a) Making use of (1), verify that $① - 1 > 1, 2, 3, 4, \dots$ and that $① - 2 > 1, 2, 3, 4, \dots$. Generalise these results.

b) Making use of (2), explain why the following inequalities hold: $① - 1 < ①$ e $① < ① + 1$.

c) Making use of (2), compute (i) $① - ①$ and (ii) $4① - ① - 3① + 3$.

d) Making use of (2), compute $(① + 1)^2$.

Exercise 62.

a) Property (2) also applies to arithmetic with negative numbers and fractions. For instance:

$$\frac{①}{4} - \frac{①}{3} = -\frac{①}{12}; \left(\frac{1}{2}\right)^{①} = \frac{1}{2^{①}}.$$

Compute:

$$\text{(i) } \frac{3①}{4} - \frac{①}{2}; \text{ (ii) } \frac{1}{2^{①}} + \frac{1}{2^{①}}$$

b) Since $① > 1, 2, 3, 4, \dots$, verify that $1/① < 1/2, 1/3, 1/4, \dots$. This is to say that $1/①$ is a *positive infinitesimal*. Verify that $2/①$ and $3/①$ are positive infinitesimals.

9.2 Summing powers of $1/2$

Consider the sum of the first four positive powers of $\frac{1}{2}$:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}.$$

One way to compute it is to set its value equal to x and verify that:

$$2x = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1 + x - \frac{1}{16}.$$

The sum we are looking for is $x = 1 - \frac{1}{16} = 1 - \frac{1}{2^4}$.

Exercise 63.

a) Using the argument just provided, find the sum of the first n positive powers of $1/2$.

b) Compute:

$$\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{①}}.$$

c) Compute:

$$\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{①-1}}.$$

9.3 Counting subdivisions of ℕ

We can describe ℕ as the following sequence:

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad \ldots$$

If we rewrite the same sequence on two columns, moving from top to bottom and from left to right, we obtain:

$$\begin{array}{cccccc} 1 & 3 & 5 & 7 & 9 & \ldots \\ 2 & 4 & 6 & 8 & 10 & \ldots \end{array}$$

We have split ℕ into two equal parts, the odd and the even numbers. We obtain a finer subdivision, into three equal parts, by rewriting the initial sequence on three columns, moving again from top to bottom and from left to right:

$$\begin{array}{cccccc} 1 & 4 & 7 & 10 & 13 & \ldots \\ 2 & 5 & 8 & 11 & 14 & \ldots \\ 3 & 6 & 9 & 12 & 15 & \ldots \end{array}$$

Since there are ① numbers in ℕ, the subdivision into two equal parts splits it into $①/2$ even numbers and $①/2$ odd numbers. The subdivision into

three equal parts splits it into three sequences, each of which contains ①/3 numbers.

Exercise 64.

a) Using ①, we can describe ℕ thus:

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \dots \quad ①-2 \quad ①-1 \quad ①.$$

Since $①/2+①/2=①$, ① is even. By (2), $①-1$ is odd and $①-2$ is even. Thus, the sequence of even numbers in ℕ is:

$$2 \quad 4 \quad 6 \quad 8 \quad 10 \quad \dots \quad ①-4 \quad ①-2 \quad ①,$$

which cannot be as long as the full sequence ℕ, but is only half as long. Compare the sequence of the even numbers in ℕ with a completed count of ℕ.

b) Compare the sequence of the odd numbers in ℕ with a completed count of ℕ. Compare the sequence of the multiples of 3 in ℕ with a completed count of ℕ.

c) Using ①, describe the subdivision of ℕ into four equal parts. How many numbers are there in each part of this subdivision?

Exercise 65.

a) Compare a completed count of ℕ with the tenth part of ℕ containing 2.

b) Compare a completed count of ℕ with the tenth part of ℕ containing 10.

9.4 Il paradosso di Machina

Note: *some of the following exercises are starred. These exercises are to be solved by making use of ①. The other, unstarred exercises are to be solved without making use of ①.*

Emmy holds infinitely many one Euro coins[1]. Her reserve of coins is counted by a completed count of ℕ. Emmy asks her friend Felix to play a decision game following her rules. At the n-th decision, Felix may:

[1] This problem was oroginally stated in [**16**] as a simplified version of an analogous problem presented in [**5**].

1. return all coins he currently owns (possibly none) and receive or receive back the first ten coins he was given by Emmy;
2. return all coins he currently owns (possibly none) and receive the coins that, in the order in which Emmy pays them out, occupy the positions from $10n+1$ to $10(n+1)$.

We may assume that Emmy asks Felix to make the first decision in $1/2$ minutes, the second after $1/4$ more minutes, the third after $1/8$ more minutes and so on.

Exercise 66.

a) *How many decisions is Felix to make?

b) *How long does it take Felix to make the first thousand decisions? How long does it take him to make all of them?

c) *How many coins does Emmy hold?

d) Should Felix always choose action 2 over action 1?

e) *Should Felix always choose action 2 over action 1?

f) *If Felix alternates both actions, starting with 1, what is his final payoff?

Exercise 67. *If Felix starts with a choice of action 1 and, for every choice of action 1, he makes two consecutive choices of action 2, which is his last chosen action? What is his final payoff? If Emmy's coins were numbered, which numbers would the coins in Felix payoff show?

Exercise 68. Evaluate the assumptions supporting the following argument and its correctness:

> If Felix chooses action 2 all the time, his final payoff will be zero. If k is any positive integer, it must lie between $10n+1$ and $10(n+1)$ for some value of $n \geq 0$. But this implies that the k-th coin held by Emmy, even if it is paid out after Felix makes n decisions, will be returned to Emmy in the next decision. Because of this, every coin Emmy pays out is ultimately returned to here and Felix's payoff is zero.

9.5 Yablo's paradox

For any item[2] n in a completed count of $\mathbb{N}$, at $1/2^n$ minutes after noon, Emmy asks Felix to choose 1 or 0. Felix can win a chocolate bar if he manages to make his choices according to the following rule: he must choose 1 if he has always previously chosen 0 and must choose 0 otherwise (that is, if he previously chose 1 at least once).

Exercise 69. Describe Emmy's challenge when it consists of ten choices only. In this case, determine how long it takes Felix to make all the choices he is required to make and identify a strategy that he might adopt in order to win the chocolate bar.

We not turn to Emmy's challenge in its infinite version.

Exercise 70. If, when designing her challenge, Emmy had worked in ordinary numerical terms, she might have pursued the following train of thought:

> If Felix chose 1 at the n-th stage of the game, he should have chosen 0 at every earlier stage, i.e. infinitely many times before. In this case, his last choice of 0 would also follow an infinite sequence of choices of 0 and, for this reason, would violate the rule I set, which requires at least one earlier choice of 1 before 0 is chosen.

What would Emmy deduce, reasoning as above, if Felix had chosen 0 at the n-th stage of the gaem? Could, in this case, Felix respect the rule set him by Emmy?

We conclude with a numerical study of Emmy's challenge.

Exercise 71.

a) *How many choices is Felix asked to make?

b) *What time was the first choice made?

c) *How long does it take Felix to make every choice?

d) *How could one criticise Emmy's argument from exercise 71?

e) *What strategy would enable Felix to win the chocolate bar?

[2] This problem was originally stated in [**36**] and reformulated in the version used here in [**4**].

9.6 An infinite lottery

Emmy invites Felix to engage in an infinite system of bets[3], as many as are counted by a completed count of $\mathbb{N}$. Each bet depends on a lottery: a numbered ticket is drawn from an urn that contains as many tickets as are counted by a completed count of $\mathbb{N}$. Bet number n requires Emmy to draw a ticket from the urn and to pay Felix 2 Euros if the ticket shows the number n or to receive from Felix $1/2^n$ Euros if she has drawn any other ticket. Drawn tickets are always returned to the urn, so that any drawing is effected on the same collection of tickets.

In order to study this problem, we first describe a random draw from the urn.

Exercise 72.

a) *How many tickets does the urn contain?

b) *If there were 6 tickets, the probability of drawing any one of them would be $1/6$. What is the probability of one draw in Emmy's lottery?

c) *What is the probability of drawing a ticket other than that labelled by 3 in Emmy's lottery?

d) *What is the probability of drawing a ticket other than those labelled by a numeral between 1 and 100?

Felix decides to accept the bets proposed by Emmy. He thinks that the n-th bet offers a probability close to 1 of winning 2 Euros but only presents an infinitely small risk of a loss. Since each bet is favourable, Felix expects the whole, infinite system of bets to be favourable. Let us find out if he is right.

Exercise 73.

a) *Compute, for the n-th bet, the product between the probability of a loss and the sum that, in this case, is to be paid to Emmy. Summing all such products, we obtain Felix's *expected loss*. Compute it.

b) *Compute, for the n-th bet, the product between the probability of winning and the sum that, in this case, is to be paid by Emmy. Summing all such products, we obtain Felix's *expected gain*. Compute it.

[3] This problem was originally posed in [**2**]

We turn to examining whether it is true that each bet really is favourable to Felix.

Exercise 74.

a) *Consider bet number $(①/2)-1$. Find Felix's expected gain for this bet.

b) *Find Felix's expected loss for the same bet.

c) *Write the ratio between expected loss and expected gain, supposing that the expected gain is certain (has probability 1) and, using the inequality $2^{\frac{①}{2}} > 2①$, verify that the ratio is smaller than two.

d) *The conclusion obtained in (c) continues to hold for any bet following the number $\frac{①}{2}-1$. Why? What can we conclude about the number of bets favourable to Felix?

The last exercise took the inequality $2^{\frac{①}{2}} > 2①$ as a given. The interested reader may verify it in the following exercise:

Exercise 75.

a) A sequence of 0 and 1 that consists of n elements is a binary sequence of length n. There are $4 = 2^2$ binary sequences of length 2 and $8 = 2^3$ binary sequences of length 3. Verify these claims.

b) *How many binary sequences of length $①/2$ are there?

c) *Describe $2①$ binary sequences of length $①/2$ and show that there are binary sequences of the same length that differ from all those described.

10. Worksheet F: physical paradoxes

10.1 Counting $\mathbb{N}$

Let us suppose that five objects are given to us and that we wish to count them. As we count them, we assign to them the symbols $1,2,3,4,5$. These numerical symbols are not the only ones at our disposal, but belong to the sequence $1,2,3,4,5,\ldots$, which we can extend as far as we please. We may thus describe a count of five items as follows:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\ 1 & 2 & 3 & 4 & 5. & & & \end{array}$$

A completed count is nothing but the application of an initial segment of a counting system to a collection of items. We extend this idea to infinite collections. In particular, we want to describe a completed count of $\mathbb{N}$, the collection of all items $1,2,3,\ldots$ in such a way that it can be carried out exactly as the count of five objects we just described can. **To this end, we introduce a new numerical notation that enables us to express a completed count of $\mathbb{N}$ as a specific initial segment of a counting system**. The latter completed count ends with ① (*gross-one*), which:

1. follows every number expressed in the ordinary (base ten) notation. Thus ① $> 1,2,3,4,5,6,\ldots$;
2. behaves, from the point of view of arithmetical calculations, exactly like the numbers we are used to dealing with.

A completed count of positive integers looks like this:

$$\begin{array}{ccccccccccccc} 1 & 2 & 3 & 4 & 5 & \dots & ①-2 & ①-1 & ① & ①+1 & ①+2 & \dots \\ 1 & 2 & 3 & 4 & 5 & \dots & ①-2 & ①-1 & ①. \end{array}$$

The new numerical notation we have introduced allows us to identify:

$$1, 2, 3, \dots, ① - 1, ①$$

as the initial segment of a more extensive system of numerical symbols.

Exercise 76.

a) Making use of (1), verify that $① - 1 > 1, 2, 3, 4, \dots$ and that $① - 2 > 1, 2, 3, 4, \dots$. Generalise these results.

b) Making use of (2), explain why the following inequalities hold: $① - 1 < ①$ e $① < ① + 1$.

c) Making use of (2), compute (i) $① - ①$ and (ii) $4① - ① - 3① + 3$.

d) Making use of (2), compute $(① + 1)^2$.

10.2 Summing powers of $1/2$

Consider the sum of the first four positive powers of $\frac{1}{2}$:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}.$$

One way to compute it is to set its value equal to x and verify that:

$$2x = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1 + x - \frac{1}{16}.$$

The sum we are looking for is $x = 1 - \frac{1}{16} = 1 - \frac{1}{2^4}$.

Exercise 77.

a) Using the argument just provided, find the sum of the first n positive powers of $1/2$.

b) Compute:

$$\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{①}}.$$

c) Compute:

$$\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{①-1}}.$$

10.3 Kinetic energy

Note: *some of the following exercises are starred. These exercises are to be solved by making use of ①. The other, unstarred exercises are to be solved without making use of ①.*

Consider a one metre long track[1], whose endpoints are labelled by 0 and 1 respectively. Along the track, there lie as many mass particles as there are items in a completed count of $\mathbb{N}$. Each particle has the same mass m. We call the particles $P_1, P_2, P_3, \ldots$ and refer to their positions along the track as $X_1, X_2, X_3, \ldots$. The position of P_n is X_n, $1/2^n$ metres away from 0. Starting from 1, suppose that particle P_0 moves at the constant velocity v towards P_1 until it hits it, thus starting a series of inelastic collisions. After the first collision, P_0 is at rest at $1/2$ metres from 1, whereas P_1 moves towards P_2 at constant velocity v. After hitting P_2, P_1 is at rest and P_2 is set in motion at the velocity v. The next inelastic collisions affect $P_3, P_4, \ldots$. Suppose that the collisions have been so scheduled that the one between P_0 and P_1, occurs $1/2$ minutes after a fixed initial time, the next collision after another $1/4$ minutes, the third after an additional $1/8$ minutes, and so on.

Exercise 78. Evaluate the following argument:

> While P_0 is moving, the total kinetic energy of the particle system along the track is $(1/2)mv^2$. After P_0 collides with P_1, it comes to a halt in a state of rest, but P_1 is in motion and the total kinetic energy of the particle system is conserved. However, after one minute, every collision has taken place. Thus, every particle is at rest: in this case, the total kinetic energy of the system is zero. Kinetic energy is not conserved in the given infinite system!

[1] This problem is discussed in [**11**] and [**12**].

Exercise 79. Clarify the following remark in light of the previous exercise:

> The physical laws governing classical, inelastic collisions do not depend on the direction of time, so we can think of playing the sequence of collisions along the track in reverse. In this case we see that an infinite system at rest can spontaneously set itself into motion.

We are now ready for a numerical study of the physical problem.

Exercise 80.

a) *Specify the number of particles in the system on the track.

b) *How long does it take for all collisions to have occurred?

c) *Is there a way of scheduling the collisions so that they are completed in exactly one minute.

d) *Where is the $(① - 1)$-th particle. In which collisions is this particle involved and when?

e) *Determine the kinetic energy of the system after all collisions have taken place.

10.4 Creation ex nihilo

Consider a one metre long track[2] whose endpoints are labelled by 0 and 1. The track is marked by as many positions $X_1, X_2, \dots$ as there are items in a completed count of $\mathbb{N}$: in particular, position X_n is at a distance of $1/2^n$ metres from 0. We now introduce a rule to insert a mass particle on the track, depending on whether or not we detect a mass particle at position X_n and at time T_n. Here T_n is the instant occurring $1/2^n$ minutes after some fixed, initial instant T_0. If we insert a mass particle P_n, we take it from an infinite supply that contains as many items as make up a completed count of $\mathbb{N}$. The insertion rule considers three possible scenarios:

[2] This problem was originally presented in [**13**].

1. A mass particle is detected in X_n at T_n and moves at the constant velocity 1m/s from 0 towards 1.
2. A mass particle is detected in X_n at T_n and moves from 0 towards 1, but not at constant velocity;
3. No mass particle is detected in X_n at T_n.

Each scenario determines a sub-case of the rule:

1. In scenario 1, no insertion is made and the mass particle that was detected continues to move.
2. In scenario 2, the mass particle detected is annihilated after $1/2^{n+1}$ minutes, and the mass particle P_n, moving at the constant velocity 1m/s, is inserted beyond X_n in the direction of 1, more precisely, $1/2^{n+1}$ metres after X_n.
3. In scenario 3 the insertion is performed as in scenario 2, but without any mass particle being annihilated.

Exercise 81.

a) Frame an argument to show that, if the insertion of P_n had been made, no insertion of P_{n+1} could have been made.

b) Use this argument to conclude that no insertion of a mass particle can be made.

c) Deduce that, since no insertion is possible, there must be some particle Q moving from 0 towards 1 at constant velocity $1m/s$.

d) The mass particle Q cannot be any one of the P_n. Why? Conclude that Q is created ex nihilo.

We conclude with a numerical treatment of the insertion problem.

Exercise 82.

a) *How many detections have to be effected in order to decide whether an insertion should be made or not?

b) *When and where is the first detection effected?

c) *What happens after the first detection and how does this affect the subsequent ones?

d) *Identify the missing information whose absence makes it possible to set up the argument from Exercise 83.

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