First Steps in the Arithmetic of Infinity

a Workbook with Applications to Mathematical Paradoxes

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Contents

Preface

The ideas introduced in this booklet have originated from the work of Yaroslav Sergeyev: they all revolve around the introduction of a more powerful numerical notation than the traditional ones. For example, if we use decimal notation, we can express numbers that are as large as we please, but we cannot express the size of the set of all natural numbers $\mathbb{N}$ as a number. In Sergeyev's notation this is possible: the number in question is expressed by the infinite unit ① (called *grossone*) and we can carry out calculations with it in the same way in which we carry out ordinary calculations. With ① at our disposal, we are able to tackle many old and new problems in pure and applied mathematics. Sergeyev's ideas have already found applications to the calculus, ordinary differential equations, engineering, computer science and biology. Although a broad mathematical knowledge is required to understand the applications, there are much simpler ones, which can be tackled with only a basic knowledge of the arithmetic of ① and lead to easy solutions of several paradoxes of infinity. The purpose of this exercise-based booklet is to enable the reader quickly to master the basic arithmetic of ① and to devise elementary but fascinating applications of this knowledge to several puzzles that have arisen in philosophy, decision theory and basic physics.

1. Arithmetic with an infinite unit

1.1 The infinite unit ①

Consider the set of natural numbers minus zero, namely; $\mathbb{N} = \{1,2,3,4,5,6,\ldots\}$. If we collect together the first three elements we obtain $\{1,2,3\}$, where the largest number also measures the size of the set. This is true of the first four, five, $\ldots$, n elements, for any number that we can express using the symbols $0,1,2,\ldots,9$. We shall say that any such number is *expressible in decimal form*. We now assume that there is a measure ① for the whole set $\mathbb{N}$. In other words, **we assume that ① > 1, ① > 2, ① $> 3, \ldots$, i.e. that ① is bigger than any number n that we can express in decimal form**. **We do not regard ① as a new number, but rather as a natural number that we could not describe in the familiar decimal form**. Using the numerical symbol ① (called *grossone*) we can write the set of natural numbers in the following, new way:

$$\mathbb{N} = \{1,2,3,\ldots,① - 2, ① - 1, ①\}$$

Note that, for every number n expressible in decimal form, ① $> n$ and ① $> n+1$, since $n+1$ is also expressible in decimal form. On the other hand, ① is a new symbol, not expressible in decimal form. So, even if we can express ① $+ 1$ in the new notation, we *do not* take it to be a natural number, since the largest natural number is ① and ① $+ 1 >$ ①. If you are familiar with the concept of a real number, you may take ① $+ 1$ to be an infinitely large real number.

We assume that **calculations with the numerical symbol ① can be carried out in the usual way.** In particular, the equalities: ① $-$ ① $= 0$ and $0$① $= 0$ hold (multiplication is denoted by juxtaposition, i.e. 'xy' means 'x times y').

■ **Example 1.1** *Simplify the expression* ① $+ 2$① $+ 3$. Since ① $+ 2$① $= 3$①, we can calculate as follows:

$$\begin{aligned} ① + 2① + 3 &= (① + 2①) + 3 \\ &= 3① + 3 \\ &= 3(① + 1) \end{aligned}$$

■

■ **Example 1.2** *Simplify the expression* $5(①+2)-4(①+3)$. We proceed as follows:

$$\begin{aligned} 5(①+2)-4(①+3) &= (5①+10)-4(①+3) \\ &= (5①+10)-4①-12 \\ &= 5①+10-4①-12 \\ &= 5①-4①+10-12 \\ &= ①-2 \end{aligned}$$

■

Exercise 1.1 Simplify the following expressions:

a) $①+4①+3$;
b) $3①-2(①-3)$;
c) $5(①+4)-4(①+5)$;
d) $24(①+3)-8(9+3①)$;
e) $10(6(3①-4)-2(4①+7))$.

■

We can carry out calculations with ① even when positive and negative numbers are involved. For example, $2①-3① = -①$ and $①(-2) = -2①$. We also calculate powers in the usual way. In particular, the following equalities hold:

$$0^{①} = 0,\ ①^0 = 1,\ 1^{①} = 1.$$

■ **Example 1.3** *Simplify the expression* $3①(2+3①)-①(3-①)$. We proceed as follows:

$$\begin{aligned} 3①(2+3①)-①(3-①) &= (6①+9①^2)-①(3-①) \\ &= (6①+9①^2)-3①+①^2 \\ &= 6①+9①^2-3①+①^2 \\ &= 10①^2+3① \\ &= ①(10①+3) \end{aligned}$$

■

Exercise 1.2 Simplify the following expressions:

a) $①(2+①)-4①-3①(1+①)$;
b) $①[3(①^2-①+4)-6①(①-1)-12]$;

c) $2[①(2①+7)+2(①^2+1)-①(①^2+4①-11)]$;

d) $(①+1)^3-①(3①+3)$;

e) $2-2①+[(①+1)(①-1)]$.

Finally, we want to use the new symbol ① in calculations involving fractions. Again, the usual rules of calculation apply. In particular, the following equalities hold:

$$\frac{①}{①}=1,\ \frac{0}{①}=0.$$

Example 1.4 *Simplify the expression* $\frac{①}{2}-\frac{3(①+1)}{3}$. We proceed as follows:

$$\begin{aligned}\frac{①}{2}-\frac{3(①+1)}{3} &= \frac{3①-6(①+1)}{6}\\ &= \frac{3①-6①-6}{6}\\ &= \frac{-3①-6}{6}\\ &= -\frac{①+2}{2}\\ &= -\frac{①}{2}-1\end{aligned}$$

Exercise 1.3 Simplify the following expressions:

a) $\frac{1}{7}\left(\frac{①^2}{2}+\frac{①^2}{3}-\frac{①^2}{4}\right)$;

b) $\left[\left(\frac{①}{4}+\frac{1-①}{2①}\right)-\frac{①-2}{4}\right]$;

c) $\left(\frac{①}{3}-\frac{①}{4}\right)\left(\frac{3}{①}+6+3①\right)$;

d) $\left[\left(\frac{1}{①}+\frac{1}{①^2}-\frac{1-①^2}{①^3}\right)-\frac{3}{①}\left(\frac{1}{①}-\frac{3+①}{3①}\right)\right]$;

e) $-\frac{1}{3}\left[\left(\frac{4①^2-3①-1}{①^2}\right)\left(\frac{①}{①-1}-\frac{4①}{4①+1}\right)\right]$.

1.2 Sizes of parts of $\mathbb{N}$

The number ① measures the size of $\mathbb{N}$. We would like to say that, if we take away from $\mathbb{N}$ all of the odd numbers, then we are left with half of ① numbers, i.e. we are left with $\frac{①}{2}$

numbers. The same should happen if we take away all even numbers. In order to obtain this conclusion, and to be able to evaluate the size of certain parts of $\mathbb{N}$, we assume the following principle:

If k,n are numbers expressed in decimal form and $1 \leq k \leq n$, then we say that the set $\mathbb{N}_{k,n} = \{k, k+n, k+2n, k+3n \ldots\}$, which is a part of $\mathbb{N}$, has $\frac{①}{n}$ elements. Since a set is a collection of objects, $\frac{①}{n}$ is a natural number that counts them.

■ **Example 1.5** *Find the size of the set of odd numbers*. To this end, we appeal to the principle just stated and note that, when $k = 1$ and $n = 2$, the set $\mathbb{N}_{1,2}$ is exactly the set $\{1,3,5,7,\ldots\}$ of all odd numbers. It then follows that this set has size $\frac{①}{2}$. ■

■ **Example 1.6** *Find the size of the set of multiples of* 5. To this end, we appeal to the principle just stated and note that, when $k = 5$ and $n = 5$, the set $\mathbb{N}_{5,5}$ is exactly the set $\{5,10,15,20,\ldots\}$ of all multiples of 5. It then follows that this set has size $\frac{①}{5}$. ■

Exercise 1.4

a) Verify that $\mathbb{N}_{2,2}$ is the set of even numbers.

b) Write down the first five elements of $\mathbb{N}_{3,5}, \mathbb{N}_{4,5}$, $\mathbb{N}_{6,7}$ and $\mathbb{N}_{7,7}$.

c) What values of k,n determine the set of multiples of 3? What is the size of this set?

■

Exercise 1.5

a) What values of k,n determine $\{7,28,49,70,\ldots\}$? What is the size of this set? Give two more examples of sets that have its size.

b) What values of k,n determine $\mathbb{N}$ itself?

c) Find the size of the set that contains the simultaneous multiples of 4 and 6.

d) What is the size of the set that contains the multiples of 5 as well as the multiples of 7, but not the common multiples of 5 and 7?

■

Exercise 1.6

a) What values of k,n determine the set of predecessors of multiples of 4. What is the size of this set?

b) What is the size of the set of natural numbers whose last digit is 0?

c) Find the size of the set that contains only the odd multiples of 3.

■

As with ①, we regard numbers like $\frac{①}{2}, \frac{①}{3}, \frac{①}{4}, \ldots$ not as new numbers but simply as natural numbers that cannot be expressed in decimal form. **If we restrict attention to decimal forms only, we can only express the sizes of finite parts of $\mathbb{N}$. With the infinite unit ①, we can also express the sizes of some among its infinite parts**.

1.3 Even and odd numbers

An integer number is even if, and only if, it is a multiple of 2. Otherwise it is odd. We have seen in section 2 that $\frac{①}{2}$ is a natural number. Thus, $① = 2\frac{①}{2}$ is even. The difference of two natural numbers is even either when they are both even or when they are both odd: since ① is even and 1 is odd, $① - 1$ is odd. Its predecessor $① - 2$, on the other hand, is even.

■ **Example 1.7** *Is* $(① - 5)①$ *even or odd? Is it a natural number?* Since ① is even and $① - 5$ is odd, their product $①^2 - 5①$ is even. This number is whole but it is not in $\mathbb{N}$ because $①^2 = ①① > 6①$, so that $①^2 - 5① = ①① - 5① > 6① - 5① = ①$. Since ① is the largest natural number and $①^2 - 5①$ is larger, it cannot be in $\mathbb{N}$. Consequently, it is not a natural number. ■

■ **Example 1.8** *Is* $①^2 - 5① - 1$ *even or odd?* Since $①^2 - 5①$ was even, its predecessor $①^2 - 5① - 1$ must be odd. Alternatively, we could have argued that, since $①^2 - 5①$ is even and, thus, an integer multiple of 2, for the original number to be even, 1 should also be an integer multiple of 2. But this is impossible, so the number in question is odd. ■

Exercise 1.7

a) Is $\frac{1}{3}①$ even or odd?

b) Is $\frac{①}{7}$ even or odd?

■

Exercise 1.8 Find the odd numbers in the following list:

a) $\frac{①}{5}$;

b) $\frac{①}{3} + 1$;

c) $① - \frac{①}{6}$;

d) $\frac{①}{3} + \frac{①}{4} - 2$;

e) $3\left(3 + \frac{①}{6}\right)$.

f) $\left(\frac{①}{5} - \frac{①}{7}\right) + \frac{①}{2}$;

g) $\left(\frac{①}{3} + 1\right)\left(\frac{①}{3} - 1\right)$;

h) $\frac{①+3}{3} + \frac{①}{2}$;

i) $\frac{①+4}{2} + \frac{①+6}{3}$;

■

1.4 Geometric Series

Consider the following sequence of numbers:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$$

It is usually assumed that there are as many terms in this sequence as there are numbers in $\mathbb{N}$. This means that the sequence has ① terms.

Exercise 1.9 Write the last three terms of the sequence.

The summation of all ① terms in the sequence is called the *geometric series of ratio* $1/2$. Call this geometric series S.

Exercise 1.10

a) Verify that S satisfies the equality: $2S = 1 + S - \dfrac{1}{2^{①}}$.

b) In view of the last exercise, the sum of the geometric series of ratio $1/2$ is $1 - \frac{1}{2^{①}}$. Show that $\frac{1}{2^{①}}$ is smaller than $1/2, 1/4, 1/8$.

c) Can you show that, if n is a natural number that can be expressed in decimal form, then $\frac{1}{2^{①}} < \frac{1}{n}$ holds?

Example 1.9 *Compute the sum of the geometric series* $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$. The whole series can be rewritten as follows:

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{3^{①-1}} + \frac{1}{3^{①}}.$$

Calling this series S, it suffices to set up an equation similar to the one used in Exercise 1.10-(a), this time involving $3S$ instead of S. Solving this equation for S yields the desired sum.

Exercise 1.11

a) Compute the sum of the geometric series $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$.

b) Compute the sum of the geometric series $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots$.

c) Compute the sum of the geometric series $\frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^3} + \ldots$.

d) Consider the series $a + a^2 + a^3 + \ldots$, with $a > 0$, and suppose that it contains exactly ① terms.
   - Write the last three term of the series.
   - Calling its sum S, set up an equation in the unknown S similar to that in exercise 1.10-(a) (*Hint*: divide the series by a term by term).
   - Using the equation you set up, find a value for S when $a = 1/6$, when $a = 3$ and when $a = ①/2$ respectively.

Consider again the geometric series or ratio $1/2$. This series can be rewritten as:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots$$

where terms being summed are the first, second, third, ..., ①-th power of the same quantity $1/2$. In order to convey this fact, a much more compact notation can be used to describe the series, namely:

$$\sum_{i=1}^{①} \frac{1}{2^i}.$$

where the Greek letter Σ ('sigma') is a summation symbol and the index i below it is made to range from the first term of the summation (the 1-st power of $1/2$) to its last term (the ①-th power of $1/2$).

■ **Example 1.10** *Compute the following summation:* $\sum_{i=1}^{①-3} \frac{1}{4^i}$. First, let us compute the sum of the geometric series of ratio $1/4$, namely:

$$S = \sum_{i=1}^{①} \frac{1}{4^i}.$$

Since $4S = 1 + S - \frac{1}{4^{①}}$, we find that:

$$S = \frac{1}{3}\left(1 - \frac{1}{4^{①}}\right) = \frac{1}{3}\left(\frac{4^{①} - 1}{4^{①}}\right).$$

In order to compute the value of the original summation, one can apply exactly the same technique, but letting $\frac{1}{4^{①-3}}$ play the role that in the series with ① terms is played by $\frac{1}{4^{①}}$. An alternative, long-winded procedure consists in subtracting from the sum of the series with ① terms the last three terms. If we pursue this strategy, we obtain:

$$\sum_{i=1}^{①-3} \frac{1}{4^i} = \frac{1}{3}\left(\frac{4^{①} - 1}{4^{①}}\right) - \left(\frac{1}{4^{①-2}} + \frac{1}{4^{①-1}} + \frac{1}{4^{①}}\right).$$

We can simplify the last expression as follows:

$$\begin{aligned}
\sum_{i=1}^{①-3} \frac{1}{4^i} &= \frac{1}{3}\left(\frac{4^{①} - 1}{4^{①}}\right) - \left(\frac{1}{4^{①-2}} + \frac{1}{4^{①-1}} + \frac{1}{4^{①}}\right) \\
&= \frac{1}{4^{①-2}}\left[\frac{4^{①} - 1}{48} - \left(1 + \frac{1}{4} + \frac{1}{16}\right)\right] \\
&= \frac{1}{4^{①-2}}\left[\frac{4^{①} - 1}{48} - \frac{63}{48}\right] \\
&= \frac{1}{4^{①-2}}\left[\frac{4^{①} - 64}{48}\right] \\
&= \frac{1}{4^{①-2}}\left[\frac{4^{①-2} - 4}{3}\right] \\
&= \frac{1}{4^{①-3}}\left[\frac{4^{①-3} - 1}{3}\right]
\end{aligned}$$

■

Exercise 1.12
Compute the following infinitely long summations:

a) $\sum_{i=1}^{①-2} \frac{1}{2^i}$;

b) $\sum_{i=1}^{①-1} \frac{1}{3^i}$;

c) $\sum_{i=1}^{①-4} \frac{1}{5^i}$;

d) $\sum_{i=1}^{\frac{①}{2}} \frac{1}{2^i}$;

e) $\sum_{i=1}^{\frac{①}{2}-1} \frac{1}{2^i}$;

f) $\sum_{i=1}^{\frac{①}{3}-2} \frac{1}{2^i}$;

■

1.4.1 Supplementary material: infinite series

The geometric series of ratio $1/2$ can be specified by declaring that the n-th summand of the series is $1/2^n$. This rule allows one to write the any term of a series, once we specify a numerical value for n. For instance, if $n = 3$, the value is $1/2^3$ and, in a similar manner, if $n = ①$, then the value is $1/2^{①}$. If we now turn to the expression:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n},$$

we are able to conclude that this summation may be finite or infinitely long, depending on the value we choos to specify for n. If $n = 6$, the summation is finite, whereas, if $n = ① - 1$ or $① + 5$, the summation is infinitely long. Since we are also able to express the value of the sum as a general term, i.e., $1 - 1/2^n$, this term yields the sum of a finite or an infinitely geometric progression of ration $1/2$, depending on the value we choose for n. The same observations apply to other types of summation. If we are able to define the n-th summand, we can specify a, finite or infinite, length of the summation and, if we are also able to write the sum as a general term, we can easily compute it when n is finite or infinitely large. Consider for instance the summation whose n-th term is n. It may be written as:

$$1 + 2 + 3 + 4 + \ldots + n.$$

By considering the array:

$$\begin{array}{ccccccccccccc} 1 & + & 2 & + & 3 & + & 4 & + & \ldots & + & n \\ n & + & n-1 & + & n-2 & + & n-3 & + & \ldots & + & 1 \end{array}$$

it is easy to realise that $1 + 2 + \ldots + n$ can be obtained by summing all n columns and dividing the result by 2, which yields:

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

■ **Example 1.11** *Compute the sum of all natural numbers.* The sum of all natural numbers is $1 + 2 + 3 + \ldots + n$ when $n = ①$. Using the equality just obtained we can conclude that the value of this sum is:

$$1 + 2 + 3 + \ldots + ① - 1 + ① = \frac{①(① + 1)}{2}$$

■

Exercise 1.13

a) Find the sum of the first 35 natural numbers.

b) Find the sum of the first ①/2 natural numbers.

c) Find the sum of the first ①/3 natural numbers.

■

If we restrict attention to even numbers only, we may express the summation of the first n even numbers as a general term by considering the array:

$$\begin{array}{ccccccccccccc} 2 & + & 4 & + & 6 & + & 8 & + & \dots & + & 2n \\ 2n & + & 2n-2 & + & 2n-4 & + & 2n-6 & + & \dots & + & 2, \end{array}$$

from which it is possible to deduce that the sum of the first n even numbers is:

$$2+4+6+8+\ldots+2n = \frac{(2n+2)n}{2} = n(n+1)$$

■ **Example 1.12** *Compute the sum of all even natural numbers*. In order to do so, we must bear in mind that there are ①/2 even numbers (see section 1.2). When $n =$ ①/2, the general term for the sum of the first n even numbers becomes:

$$2+4+6+\ldots+① -2+① = \frac{①}{2}\left(\frac{①}{2}+1\right) = \frac{①(①+2)}{4}.$$

■

It is now possible to find the sum of all odd natural numbers by computing the difference between the sum of all natural numbers and the sum of all even natural numbers.

■ **Example 1.13** *Compute the sum of all odd natural numbers*. The calculation we must perform is:

$$\frac{①(①+1)}{2} - \frac{①(①+2)}{4} = \frac{①(2①+2)-①(①+2)}{4} = \frac{①^2}{4}.$$

■

Exercise 1.14

a) Consider the sum of the first ① even numbers. What are the last three terms of this infinitely long summation? Are they natural numbers?

b) Compute the sum of the first ① even numbers.

c) The sum of the first two odd numbers is 4. The sum of the first three is 9. Find the sum of the first four and of the first five.

d) Can you write a general term for the sum of the first n odd numbers? Use this term to find the sum of all odd natural numbers.

e) Find the sum of the first ① odd numbers.

f) It is not necessary to restrict attention to summations that contain as many as ① terms. It is possible to apply the approach developed so far to summations with more than ① terms. Bearing this in mind, compute the sum of all numbers from 1 to ①+4 and the sum of all numbers from 1 to ①+①.

Example 1.14 *Compute the sum* $1-2+3-4+5-\ldots+①-1-①$. One way of doing it is to split the whole summation, which has ① terms, into two summations of $①/2$ terms each. One of the two summations is the sum of all odd natural numbers and the other is -1 times the sum of all even natural numbers. Thus (verify the details of the calculation below):

$$1-2+3-4+5-\ldots+①-1-① = \frac{①^2}{4}-\frac{①(①+2)}{4} = -\frac{2①}{4} = -\frac{①}{2}.$$

Exercise 1.15

a) Is there an easier method of computing the last sum? (*Hint*: consider placing brackets around pairs of terms. How many pairs are there?)

b) Compute the sum $2-1+4-3+6-5+\ldots+①-(①-1)$. Can you do it in two different ways?

Example 1.15 *Compute the sum* $2-4+6-8+10-\ldots+(①-2)-①$. This summation can be found in an easy or in a more long-winded way. Both are worth exploring. In the easy way, we just place brackets as follows:

$$(2-4)+(6-8)+(10-12)+\ldots+((①-2)-①).$$

Each term within brackets simplifies to -2 and, since there are $①/4$ terms (verify this!), the sum is $-①/2$. In the long-winded way, we may observe that the given summation has as many terms as there are even natural numbers, i.e., it has $①/2$ terms. There are $①/4$ positive terms, whose sum is:

$$2\left(1+3+5+\ldots+\left(\frac{①}{4}+1\right)\right) = 2\left(\frac{①^2}{16}\right) = \frac{①^2}{8},$$

which is calculated as twice the sum of the first $①/4$ odd natural numbers. The negative terms are the first $①/4$ multiples of 4. Their sum can be written as:

$$-4\left(1+2+3+\ldots+\frac{①}{4}\right) = -\frac{①^2+4①}{8}.$$

Exercise 1.16

a) Verify the details in the calculation of the sum of negative terms from the last example.

b) Verify that the long-winded calculation leads to the same result as the quicker calculation.

c) Compute the sum $1-2+3-4+\ldots-①+(①+1)-(①+2)+(①+3)$.

Without appealing to the infinite unit ①, the summations computed in this section can only be described by assigning to them the symbol $+\infty$ or the symbol $-\infty$. In other words, they are indistinguishable but for their sign. The introduction of ① affords far more accurate evaluations: e.g. it allows to tell whether certain infinitely long summations have an even or odd value or to compare the values of two infinitely long summations or even, finally, to find out whether the value of one of them is a (finite or infinite) multiple of the other. This is a simple but important instance of the remarkable increase in computational power that the adoption of a numerical notation based on the infinite unit ① makes available. Much more than what has been described here can be done. For instance, one could consider numbers like $①^{-1}$, which are smaller than any positive number expressible in decimal form and perform infinitely long summations with them or their multiples, which may yield finite, infinitely small or infinitely large values. Examples of these phenomena are discussed in Sergeyev (2009).

Notes

The arithmetic of infinity, whose basic ideas have been described in this chapter, is the creation of Yaroslav Sergeyev. An accessible, more extensive development of his ideas than that offered in this chapter can be found in Sergeyev (2003, 2008, 2009, 2010).

2. Solving Paradoxes

2.1 Philosophical Paradoxes

The techniques introduced in Chapter 1 can be used to compute the resolutions of several paradoxes of infinity that prove intractable in the absence of an infinite unit measuring the size of $\mathbb{N}$. They can also be used to construct more paradoxes or variations of known paradoxes, and to resolve these. We shall encounter several types of paradoxes of infinity: in this section we deal with philosophical paradoxes.

2.1.1 Thomson's Lamp

A lamp, which is initially switched off, is switched on after $1/2$ minutes, off after $1/2+1/4$ minutes, then on again after $1/2+1/4+1/8$ minutes, and so on. We assume that the lamp is switched on and off as many times as there are numbers in $\mathbb{N}$.
Let us first try to describe Thomson's lamp without appealing to ①. The sequence of switchings may be represented thus:

$$\mathsf{on}_1, \mathsf{off}_2, \mathsf{on}_3, \mathsf{off}_4 \ldots.$$

For each n expressible in decimal form, the n-th switching has occurred before 1 minute.

■ **Example 2.1** *Show that the second switching is performed before* 1 *minute.* The second switching is performed after $1/2+1/4=3/4$ minutes and $3/4<1$. ■

Exercise 2.1

a) How long does it take to switch the lamp on and off 4 times?

b) How long does it take to switch the lamp on and off 1000 times?

c) Show that finitely many switchings take less than 1 minute.

Although we can answer all of the previous questions without using ①, there is one question which seems really hard to answer if we do not appeal to it. The question is: when 1 minute has passed, is the lamp on or off?

Exercise 2.2 Consider the following argument:

> If the lamp is on before one minute has passed, then at a later time, before one minute has passed, the lamp is turned off. On the other hand, if the lamp is off before one minute has passed, then at a later time, before one minute, the lamp is turned on. Therefore, after one minute has passed, the lamp is neither on nor off, since each one of these states is turned into the other before one minute has passed.

Discuss this argument without appealing to ①: is it convincing? ▪

Now let us tackle Thomson's lamp using the infinite unit ①. In this case, we are in a position to compute how long it takes to switch the lamp on and off infinitely many times.

▪ **Example 2.2** *How long does it take to switch the lamp on and off ① times?* We simply have to compute the sum of the geometric series:

$$\sum_{i=1}^{①} \frac{1}{2^i}.$$

We already know that the sum of this series is $1 - \frac{1}{2^{①}}$. Thus, it takes a little less than one minute to switch the lamp on and off ① times. ▪

Exercise 2.3

a) How long does it take to switch the lamp on and off $① - 1$ times?

b) How long does it take to switch the lamp on and off $\frac{①}{3}$ times?

c) If the lamp is switched on after $1/3$ minutes, then off after $1/9$ more minutes, then on again after $1/27$ more minutes, how long does it take to switch it on and off $\frac{①}{2}$ times?

▪

Note that, **whereas, without relying on ①, we could only specify that the lamp was switched on and off infinitely many times, now we can specify a number that counts a sequence of infinitely many switchings**. The question whether the lamp is on or off after infinitely many switchings can now be addressed, because the number of actions performed on the switch can be specified.

▪ **Example 2.3** *Is the lamp on or off after 12 switchings?* In the sequence of switchings, the lamp is on at the odd-numbered places and off at the even-numbered places. Since 12 is even, it is off after 12 switchings. ▪

Exercise 2.4

a) Is the lamp on or off after ① switchings?

b) Is the lamp on after $\frac{①}{5}$ switchings?

c) Is the lamp off after $\frac{①-6}{2}$ switchings?

d) Is the lamp on after $\frac{2①-2}{3} - \frac{①}{2}$ switchings? ■

The difficulty posed by Thomson's lamp resides in the fact that the question of finding its state after 'infinitely many switchings' is too vague to be answered, since the number of switchings is not being specified. This situation is similar to one in which we may be asked whether the lamp is on or off after finitely many switchings: without knowing the number of switchings, an answer proves equally elusive. The infinite unit ① makes it possible to supply the missing information: if a count of infinitely many switches can be expressed, then it is easy to find out what the last state of the lamp will be.

2.1.2 Arsenjevic's Cube

A builder is piling red and green slabs on top of one another, in an alternating fashion. The first slab is red and $1/2$ inches thick. It is placed on the ground after $1/2$ minutes from a given initial time. The second slab, green and $1/4$ inches thick, is placed on top of the red slab after $1/2+1/4$ minutes have passed. The third slab, red and $1/8$ inches thick, is placed on top of the first two after $1/2+1/4+1/8$ minutes have passed. This process goes on to infinity, after the same fashion as Thomson's lamp: the slabs available are as many as there are numbers in $\mathbb{N}$.

Exercise 2.5 Explain in what respects the setting just described resembles Thomson's Lamp. ■

Exercise 2.6 Without appealing to ① frame an argument similar to the one given in Exercise 2.2 to argue that, after one minute, the builder has constructed a pile of infinitely many slabs whose top is neither red nor green. ■

Exercise 2.7 Appealing to ①, explain in what way the argument you have constructed is not conclusive. ■

We now continue our analysis of the building process by appealing to ①. Suppose that the builder has piled up ① slabs. Then we may ask, and answer, a few questions concerning this pile.

■ **Example 2.4** *If the pile has ① layers, what colour is the top layer?* Since the first layer is red, if one counts the layers from bottom to top, the even layers in the count are green. Because ① is even, the top of the pile is green. ■

Exercise 2.8

a) What colour is the top of a pile with $\frac{①}{7} - 3$ layers?

b) What colour is the top of a pile with ① + 2 layers? How long does it take to complete this pile, at the rate at which the builder is working?

c) If ① layers have been piled up, is the resulting pile a cube of side 1 inch?

Now consider two builders: one of them acts exactly as described above, whereas the other lags behind. More precisely, the second builder places the $1/2$ inches thick slab into position when the first builder deals with the $1/4$ inches thick slab; he then places the $1/4$ inches thick slab into position when the first builder deals with the $1/16$ inches thick slab, and so on. Moreover, the first builder stops after ① operations and the second builder stop when the first does.

Exercise 2.9 Consider the following argument:

> Even if the second builder lags behind the first, when 1 minute has passed they have piled up the same sequence of slabs. To see this, note that, the $1/4$ inches thick slab is dealt with by the first builder after $1/2+1/4$ minutes, while it takes the second builder $1/2+1/4+1/8+1/16 = 15/16$ minutes to do the same, even though each builder adds it to the pile before one minute has passed. Similarly, it takes the first builder $7/8$ minutes to deal with the $1/8$ inches thick slab and the second builder will do that after $63/64$ minutes, i.e., still before one minute. But this means that, for every number n, both builders will have added to the pile the slab that is $1/n$ inches thick before one minute has passed.

Appeal to ① in order to explain in what respects this argument is inconclusive.

Example 2.5 *Has the second builder dealt with the slab that is* $1/2^{①-2}$ *inches thick when both builders stop?* The first builder has dealt with the last slab after ① operations. In the meantime, the second builder has performed only one half of the operations performed by the first builder, i.e., only $\frac{①}{2}$ operations. The second builder is thus dealing with the slab $1/2^{\frac{①}{2}}$ when the first builder, and thus the second, stops. Now it suffices to note that:

$$\frac{①}{2} + 2 < \frac{①}{2} + \frac{①}{2} = ①,$$

from which it follows that:

$$\frac{①}{2} < ① - 2 \text{ and, thus } 2^{\frac{①}{2}} < 2^{①-2}.$$

This implies that the last slab handled by the second builder is thicker than the slab that is $1/2^{①-2}$ inches thick. As a result, the second builder never handles that slab. His pile is not as tall as the first builder's.

Exercise 2.10 Suppose that there is a third builder who starts piling slabs when the first builder has dealt with the slab that is $1/16$ inches thick. The third builder then deploys his $1/4$ inches thick slab when the first builder deals with the $1/512$ inches thick slab and deploys his $1/8$ inches thick slab when the first builder deals with the $1/16344$ inches thick slab. The third builder stops when the first completes ① operations.

a) How many operations does the third builder perform?

b) How tall is the pile of slabs put together by the third builder?

c) What colour is the top of the third builder's pile of slabs?

d) If the third builder had continued after the first builder stopped, could he have piled up ① slabs within one minute?

■

2.1.3 Black's Machines

Consider two machines A and B. Machine A transfers a marble from a right tray to a left tray at increasing speed: the first transfer is performed after $1/2$ minutes, the second after $1/2+1/4$ minutes, and so on. Machine B undoes what machine A did between two successive operations of A. Thus, for instance, after $1/2$ minutes, machine A has moved the marble to the left tray and, $1/8$ minutes later, machine B will restore this marble to the right tray. The marble is going to be moved again to the left after $1/8$ minutes, i.e., once the overall time of $1/2+1/4$ minutes has elapsed.

Exercise 2.11 Consider the following argument:

> If A and B perform the same number of operations, then the position of the marble after one minute cannot be determined. This is because if, before one minute, the marble was in the left tray, an operation has been carried out, before one minute, to move it to the right tray. On the other hand, if, before one minute, the marble was in the right tray, an operation has been carried out, before one minute, to move it to the left tray.

Compare this argument to the arguments in exercise 2.2 and exercise 2.9. Appeal to ① in order to explain in what way this argument is not conclusive. ■

■ **Example 2.6** *If each of machines A, B performs ① operations, where will the marble end up?* In order to find out, we must determine how long it takes each of A, B to perform ① operations. In the case of A, it is clear (verify this) that the time taken is $1-1/2^{①}$ minutes. In the case of B, we have to bear in mind that the first operation is performed after $1/2+1/8$ minutes, the second after $1/2+1/8+1/8+1/16 = 1/2+1/4+1/16$ minutes, and so on. Thus, the n-th operation is performed after:

$$\frac{1}{2}+\ldots+\frac{1}{2^n}+\frac{1}{2^{n+2}} = 1-\frac{1}{2^n}+\frac{1}{2^{n+2}} \text{ minutes.}$$

Thus, in particular, when $n=①$ the last machine to perform an operation is B. This means that the marble ends up in the right tray. ■

Exercise 2.12

a) Does the answer to the previous example depend on the number of operations performed?

b) Assign infinitely many operations to A and B in such a way that, when all operations have been carried out, the marble is in the left tray.

■

Notes

Thomson's lamp was originally introduced in Thomson (1954). Despite not being a cube, Arsenjevic's cube is so called after the author of Arsenjevic (1989), where the paradox is presented. Black's machines have been described in Black (1951). The resolution of these paradoxes by means of the arithmetic of infinity was firstly presented in Rizza (2016b).

2.2 Paradoxes and Decisions

In the previous section we have considered paradoxical situations arising from the idea of an infinitely long sequence of operations performed within a finite amount of time. We have seen how these situations look puzzling when the means to express numerically the length of an infinite sequence are unavailable. By supplying these means, the infinite unit ① leads to an easy arithmetical solution of what looked like paradoxical situations. This situation arises whenever sequential processes are discussed. An interesting scenario is afforded by sequences of decisions or exchanges, which we shall now study.

2.2.1 Machina's Paradox

Alice owns a stack of as many dollar bills as there are numbers in $\mathbb{N}$. The bills are labelled $1,2,3,\dots$ and so on. Alice sets up a challenge for Barbara: she is supposed to make as many decisions as there are numbers in $\mathbb{N}$ and, for each one of these decisions, Alice will pay out some dollar bills to her and withdraw others. The goal for Barbara is to make her decisions in such a way that her final payoff is as large as possible. The n-th decision is a choice between one of the following actions (before the first decision is made, Barbara does not hold any dollar bills):

1. return all bills currently held and receive (or receive back) the bills labelled 1 to 100;
2. return all bills currently held and receive the new bills labelled $100n+1$ to $100(n+1)$.

Alice also asks Barbara to make the first decision in $1/2$ minutes, the second in $1/2+1/4$ minutes, the third in $1/2+1/4+1/8$ minutes and so on.

■ **Example 2.7** *What is the final payoff if Barbara always chooses action i.?* In this case Barbara keeps receiving the same batch of dollar bills. As a consequence, however many decisions she is going to make, after her last decision she will find herself in possession of one hundred dollar bills. ■

Exercise 2.13

a) Is Barbara better off after 10 decisions when she chooses action i. all the time or when she chooses action ii. all the time?

b) Is Barbara better off after 10 decisions when she chooses action i. all the time or when she chooses to alternate action i. and action ii. (e.g. by choosing action i. in the first decision, action ii. in the second, and so on)?

c) Answer the previous two questions for 20 decisions instead of 10.

■

It is clear that the answers to questions (a), (b) and (c) from exercise 2.12 must remain the same even if one considers the first $30,40,100,\dots,n$ decisions, where n is a number expressible in decimal form. So much can be deduced if we restrict ourselves to finite sequences of exchanges. Let us now appeal to ① and look at infinitely long sequences of exchanges.

Exercise 2.14 Should the answers to questions (a), (b) and (c) from exercise 2.12 stay the same even when ① decisions are made? Provide an informal answer. The details will be worked out in exercise 2.16. ■

As the next exercise shows, it is possible to set up an argument to support the conjecture that the outcome of infinitely many decisions may be very different from the outcome of finitely many decisions.

Exercise 2.15 Alice did not know about ① when she set her infinite decision task to Barbara. Thus, she argued as follows:

> If Barbara chose action ii. all the time, she should find herself without any bills after completing her infinite decision task. The reason is that, if k is any number, then it lies between $100n+1$ and $100(n+1)$ for some $n \geq 0$. But this is to say that the bill labelled by k will be returned after n decisions. So all bills will be returned in the end, because each bill is returned after finitely many decisions.

Discuss Alice's argument by appealing to ①. Do you spot any potential issues with her argument? Provide an informal answer. A rigorous analysis of the decision problem is contained in the next exercise. ■

Let us now apply ① in order to obtain information that will be useful in assessing the argument in exercise 2.15.

Exercise 2.16

a) Describe the outcome of action ii. when it is chosen in the first, second and third decision. How many new bills are handled in each decision? How many bills are handled overall, once action ii. has been chosen three times?

b) Using the fact that there as many bills as numbers in $\mathbb{N}$, i.e., exactly ① bills, show that, if Barbara chooses action ii. all the time, the largest label on the bills handled at each of her decision can be used to determine the set $\mathbb{N}_{100,100}$, whose size counts the number of decisions made by Barbara.

c) Explain why Barbara can only make ①/10 < ① decisions.

d) Deduce from the last two exercises what the final payoff of Barbara is, when she has chosen action ii. all the time.

e) Use the results from (b) to (d) to show what the problem is with the argument in exercise 2.15.

■

If follows from the previous exercise that the final payoff Barbara receives, irrespectively of whether she chooses action i. or ii. all the time, i.e., for as long as there are bills to handle. What changes is at most the set of labels on the bills she receives after her last decision.

■ **Example 2.8** *Given the final payoff from the last exercise, what labels appear on the dollar bills in the hands of Barbara?* The dollar bills Barbara receives at the ①/10-th decision are marked by the consecutive labels ① − 99 to ①. ■

Exercise 2.17 Alice learnt about ① after setting the decision task for Barbara. She now argues as follows:

> If Barbara chose action ii. all the time, she should find herself without any bills marked by a label expressible in decimal form after completing her decision task. The reason is that, if k is any number expressible in decimal form, then it lies between $100n+1$ and $100(n+1)$ for some $n \geq 0$ expressible in decimal form. But this is to say that the bill labelled by k will be returned after finitely many decisions.

Is this argument correct? Does it imply that, after all dollar bills are handled, Barbara will be left without any? ■

2.2.2 Ross' Paradox

Alice owns a stack of as many dollar bills as there are numbers in $\mathbb{N}$. The stack is packed into a box and the individual dollar bills are labelled as bill 1, bill 2, bill 3, and so on. Alice asks Barbara to remove bills 1 to 10 from the box and return bill 1 into the box in $1/2$ minutes, then to remove bills 11 to 20 and to return bill 2 into the box after $1/4$ minutes, and so on.

■ **Example 2.9** *What happens at the third stage of this process and how long does it take Barbara to complete the first three stages?* In the first stage, bills 1 to 10 are removed and bill 1 is returned. Barbara holds bills 2 to 20. In the second stage she returns 2 and takes out bills 11 to 20. Finally, in the third stage Barbara holds bills 4 to 20 and she has returned bills 1 to 3 to the box. ■

Exercise 2.18

a) What happens at the fourth stage?

b) How many dollar bills are handled before bills 1 to 10 are returned into the box? Which dollar bills does Barbara hold at that stage?

■

In this setting it is possible to set up two arguments that do not make any appeals to ① and purport to show what final outcome of Barbara's actions is going to be. The two arguments are as follows:

Exercise 2.19 Barbara reasons as follows:

> At every step in the procedure, I keep 9 dollar bills and I have to repeat the procedure infinitely many times. If I do it quickly, by one minute I'll be infinitely rich!

Is Barbara's argument compelling? ■

Exercise 2.20 Alice reasons as follows:

> For every number n, there is a stage in the procedure at which Barbara will have to return n into the box. This means that, when she has completed the procedure, the box will contain all of the dollar bills again. Barbara thinks she can become infinitely rich, but she is utterly mistaken.

Answer the following questions by appealing to ①:

1. Would it be correct to infer from Alice's argument that, once the procedure has come to an end, every dollar bill marked by a finite label has been returned to the box?
2. Is it the same to say that every dollar bill marked by a finite label has been returned and to say that every dollar bill has been returned? Answer this question by considering two scenarios: in the first scenario there are ① dollar bills but only finite labels are available; in the second scenario there are ① dollar bills and ① labels, so that every bill can be labelled.

■

Suppose that the dollar bills had not been labelled and that there were three boxes, two empty boxes A, B and a box C that initially contains the whole stack of dollar bills. If Barbara had been asked repeatedly to take ten dollar bills from box C, put one into A and nine into B, her argument to the effect that, by keeping the contents of B, she would end up infinitely rich looks more compelling. But, under this procedure, it also seems natural to conclude that A must contain as many dollar bills as were returned in the original procedure. If Alice's argument from exercise 2.20 had been correct, then it should be possible to add labels to the bills in box A and, since the labels $1, 2, 3, \ldots$ are all going to be used, it suddenly seems to follow that box B should be empty because all bills turn out to be in A. The reason why this puzzle arises is that no explicit specification of the system of numerical symbols that supports the assignment of labels to bills has been given. It may well be that it takes every label expressible in decimal form to label the dollar bills in box A, but does this mean that each dollar bill in the original stack is labelled? If we one refrains from using labels expressible using ①, maybe several dollar bills will remain unlabelled, even if all labels in decimal form are used. We shall now examine this possibility in greater detail. The largest label on a bill extracted in the first stage of the procedure is 10. The largest label extracted in the second stage is 20. The sequence of largest labels extracted at each stage can be listed as $\{10, 20, 30, \ldots\} = \mathbb{N}_{10,10}$. This part of $\mathbb{N}$ has ①/10 elements, so Barbara carries out a procedure that ends after ①/10 stages.

Exercise 2.21

a) Suppose that there are ① labels available, i.e., every dollar bill is labeled. What labels appear on the dollars handled in the last stage of the procedure?

b) One dollar bill is returned at each stage in the procedure: how many dollar bills are returned by Barbara overall and what are the labels appearing on them?

c) What is the difference between the total number of bills and the number of bills returned by Barbara?

d) Verify that we obtain the same number if we the number of bills kept at each stage by Barbara times the number of stages in the procedure. ■

2.2.3 Yablo's Paradox

For each natural number n, at $1/n$ hours past 12 pm, Alice is asked to choose either 1 or 0. If she answers according to the following rule, she will receive a chocolate. The rule is: choose 1, if you have chosen 0 at every previous round, and chose 0 otherwise (i.e. if you have chosen 1 on at least one other round.)

Exercise 2.22 Restate the game proposed to Alice when n can only be a number between 1 and 10. Show how, for this restated game, Alice could apply the rule and win a chocolate. ■

If, instead of the finite game in exercise 2.22, we consider an infinite game, we might expect complications to arise. This is, as usual, true if we restrict our expressive resources to deal with numbers expressible in decimal form only.

Exercise 2.23

a) Show that, if only numbers expressible in decimal form are used, there is no earliest time before 12pm at which Alice started playing the game.

b) If only numbers expressible in decimal form are used, the following argument can be run:

> If Alice chooses 1 at any stage n of the game, she must have chosen 0 at each previous stage. In this case, her last choice of 0 follows an infinite sequence of choices of 0 and thus it is inconsistent with the rule, which mandates an earlier choice of 1.

What happens when Alice chooses 0 at any stage n of the game? Provide an answer without making use of ① and conclude from your answer and the previous argument that the rule proposed to Alice cannot be followed.

■

We may now look at what happens when, instead of only saying that the game is played infinitely many times, we can appeal to ① and specify an infinite length for it. If the game is played as many times as there are numbers in $\mathbb{N}$, then we know that it has ① rounds.

Exercise 2.24

a) What time after 12pm is the earliest round of the game played? Find a strategy that allows Alice to win a chocolate.

b) Suppose that only $①/3+1$ rounds are played. Find the time at which the earliest round is played and describe a winning strategy.

■

Notes

Machina's paradox was introduced in Machina (2002) (this paradox is a simpler version of the harder setup described in Barrett and Arntzenius (1999). The paradox in this paper is harder to solve than Machina's but can still be handled using ①). What is often called

Ross' paradox is a problem originally presented in Littlewood (1986), which was later discussed in Ross (1988). Yablo's paradox was stated for the first time in Yablo (2000), but the particular formulation of the paradox used in this chapter is due to Bacon (2010). Machina's paradox has been originally resolved using the arithmetic of infinity in Rizza (2016b). Rizza (2015) contains a discussion and resolution of Ross' paradox.

2.3 Physical Paradoxes

So far we have seen how paradoxical situations can be constructed by taking the sequence of numbers in $\mathbb{N}$ to label a sequence of operations or a sequence of decisions. The same approach can be applied to infinite sequences of physical events to try and deduce outcomes that seem at variance with physical intuition or perhaps even physical theory, e.g., the spontaneous self-excitation of an isolated mechanical system at rest or the creation of a particle within an isolated physical system. As we shall see, such counterintuitive conclusions stem from the fact that, if there are ① physical objects but only numbers expressible in decimal form can be used to label them, certain objects cannot be labelled and seem to appear out of the blue. We offer only two instances of this setting, which are presented as projects that, based on your previous experience with this chapter, you should be able to work out through exercises.

2.3.1 Self-excitation

On an interval of unit length, whose endpoints are marked by the labels 0 and 1, as many mass particles as there are numbers in $\mathbb{N}$ occupy distinct positions. The particles in question, namely $P_1, P_2, P_3, \ldots$, all have the same mass. The position of P_n along the interval is X_n, at $1/2^n$ units of distance away from point 0. Moreover, starting from position 1, a particle P_0 of mass m and constant velocity v moves towards P_1 and triggers a series of collisions. After the first collision, P_0 remains at rest at $1/2$ units of distance to the left of position 1, while P_1 collides with particle P_2 at velocity v and halts in the position formerly occupied by P_2. The same interaction occurs along the sequence, so that every particle shifts to the next position to its left.

Exercise 2.25 Explain why all collisions will have taken place within a unit of time. ▪

Exercise 2.26 Consider the following argument:

> After one unit of time P_n has collided with the next particle P_{n+1} and come to rest. So, every particle is at rest. This means that the total initial energy of the system of particles, namely $2^{-1}mv^2$, has disappeared.

Is the conclusion of this argument correct? ▪

Exercise 2.27 Since there are ① mass particles, explain what happens to the particles $P_{①-1}$ and $P_{①}$ and when it happens. ▪

Exercise 2.28 Consider the argument in exercise 2.26. If one reversed the direction of time, the same argument could be used to show that a system at rest can self-excite, i.e., it can suddenly set itself into motion. Appeal to ① to explain why this conclusion may not follow. ■

2.3.2 Creation

Consider an interval of unit distance whose endpoints are called 0 and 1. We refer to this interval as $(0,1)$. Positions are marked on $(0,1)$ in such a way that the n-th position x_n lies $1/2^n$ units of distance from 0. Given as many identical particles P_n as there are numbers in $\mathbb{N}$, we are going to specify a rule for the insertion of P_n in the interval $(0,1)$ after $t_n = 1/2^n$ seconds from a fixed time zero. This rule can be specified by taking into account three distinct possible situations:

1. A particle is detected in x_n at t_n and it moves at unit velocity.
2. A particle is detected in x_n at t_n but it does not move at unit velocity;
3. No particle is detected in x_n at t_n.

The rule can then be spelled out as follows:

1. In case 1, nothing happens and the particle continues to move at unit velocity.
2. In case 2, it is required that the detected particle be deleted after $1/2^{n+1}$ units of time, and that the particle P_n, endowed with uniform rectilinear motion and a unit velocity, be inserted in position $x_n + 1/2^{n+1}$.
3. Case 3 is identical to case 2, except that no deletion of a particle takes place.

Exercise 2.29 Without appealing to ①, can you set up an argument to the effect that no insertion of a particle can actually be performed? *Hint*: Argue that, if an insertion of P_n had been performed, then no insertion of P_{n+1} could have been performed. ■

Exercise 2.30 Deduce from the argument set up in the previous exercise that, if no insertion can be performed, then there is a particle Q that moves at unit velocity from 0 to 1. Can Q be one of the particles P_n? Answer without appealing to ①. ■

The previous two exercises set up a train of thought whose puzzling conclusion is that Q suddenly appears into existence in the system $(0,1)$.

Exercise 2.31 Using ①, show where the first insertion may be performed. Explain why there cannot be a particle Q to appear suddenly into existence. ■

Notes

The paradoxes of self-excitation and creation were devised in Laraudogoitia (1996, 2005, 2009). They were first resolved using the arithmetic of infinity in Rizza (2016a-b).

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