Infinite Exchanges

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- When we count we go through the progression:

$$1, 2, 3, \ldots, n, \ldots,.$$

- However, often in mathematics we speak of the *entire* set $\mathbb{N}$.

- In set theory, we wish to determine and compare the sizes of sets like $\mathbb{N}$, which we call infinite.
- In order to do so we look for one-to-one correspondences as indicators of equal sizes.

- The progressions:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \dots \\ 2 & 3 & 4 & 5 & \dots \\ 3 & 6 & 9 & 12 & \dots \end{array}$$

- are assigned the same size, symbolised by $\aleph_0$. They all diverge to ∞.

- From the standpoint proposed by Yaroslav Sergeyev $\aleph_0$ is a relatively inaccurate esteem.
- Since we cannot see tails, we think of the indefinite progressions as equivalent.
- We see what we can denote by a number, so let us take ① to denote the number of elements in $\mathbb{N}$.

- We can now count the number of items in our sequences, knowing how that they must end after a specifiable number of infinitely many steps.

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & \dots & ① - 2 & ① - 1 & ① \\ \\ 2-1 & 3-1 & 4-1 & 5-1 & \dots & (① - 1) - 1 & ① - 1 & \end{array}$$

- Note that there is no one-to-one correspondence between the two sequences above. One has $① > ① - 1$ elements.

- If we take away one in every three elements along the first sequence below, we are left with ①/3 elements.

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & \dots & \frac{①}{3}-1 & \frac{①}{3} & \frac{①}{3}+1 & \dots & ①-1 & ① \\ 3 & 6 & 9 & 12 & \dots & ①-3 & ① & ①+3 & \dots & 3①-3 & 3① \end{array}$$

- There are only ①/3 multiples of 3 in $\mathbb{N}$, since anything greater than ① is not in $\mathbb{N}$.

- Note that we are NOT replacing ∞ (or $\aleph_0$) with a new symbol ①. This would not change anything.
- Instead, we take ∞ or $\aleph_0$ to collapse infinitely many distinctions, which are visible by means of ①:

$$\infty$$

$$\ldots \frac{①}{3}, \frac{①}{3} + 1 \ldots \frac{①}{2} - 1, \frac{①}{2}, \ldots, ① - 1, ①, ① + 1, ① + 2, \ldots$$

$$\aleph_0$$

- To do this is helpful because it allows us to extend the class of mathematical problems that we can treat numerically.
- In general, we obtain an expansion of the purview of numerical analysis in applications.
- The simplest cases in which this happens are puzzles that derive from inaccurate discriminations of size at infinity.

- Suppose that we have labelled a collection of ping-pong balls with the symbols $1, 2, 3, \ldots$.
- If there are as many ping-pong balls as there are numbers in $\mathbb{N}$, each available label of the form n is used, for $n \in \mathbb{N}$.
- Note: which labels we can work with depends on our *notation for numbers*, which may not include anything like ①.

- Suppose that all of our ping-pong balls are kept in a large urn.
- Stage 0: take out those with labels $1, 2, 3$, and return the ball with label 1.
- Stage 1: take out the balls with labels $4, 5, 6$ and return the one with label 2.
- Stage n: take out those with labels $3n + 1, 3n + 2, 3n + 3$ $(n \geq 0)$ and return the one with label $n + 1$.

- If we reason with actual infinity, we have taken out two ping-pong balls infinitely many times.
- We expect $2 \cdot \aleph_0 = \aleph_0$ ping-pong balls out of the urn.
- If we reason with potential infinity, we see that, at stage $n - 1$ in our procedure, we return the ball with label n.
- We do it for each finite n. We expect 0 balls out of the urn.

- Suppose that a supply of ① labels is available and that each ping-pong ball is labelled, using this supply.
- At each stage we take three distinct balls. We cannot go through ① stages, which would require 3① $>$ ① distinct balls.

- In the last stage, we have $3n + 3 = ①$, whence $n = ①/3 - 1$. There are $①/3$ stages $0, 1, 2, 3, \ldots, ①/3 - 1$.

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- At each stage, we keep two ping-pong balls. Altogether, we have taken $2(①/3)$ out of the urn.
- Clearly, $①/3$ ping-pong balls remain in the urn.

- If we had been taking and returning dollar bills, we would have faced an infinite decision problem with payoffs.
- Infinite decisions can be handled numerically if one can effect computations in base ①.
- How far can one develop the theory of utility and probability using ①? Work in progress ...

THANK YOU !